We saw in [Lec 55.2] that if $(B_t)_{t\geq 0}$ is a Brownian motion on $\mathbb{R}^d$ and $T\geq 0$, then $S_t := B_{T+t} - B_T$ is a Brownian motion on $\mathbb{R}^d$, independent from $\mathcal{F}_T \supseteq \mathcal{F}_T^B$.

Prop: Let τ be an optional time, $\nu \in \mathrm{Prob}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ s.t. $\mathbb{P}^\nu(\tau<\infty) > 0$, and define $S_t := B_{\tau+t} - B_\tau$ on $\{\tau<\infty\}$.
Then conditioned on $\{\tau<\infty\}$, $(S_t)_{t\geq 0}$ is a Brownian motion on $\mathbb{R}^d$, independent from $\mathcal{F}_\tau^+$.

To be precise: $\forall\, F \in \mathbb{B}(C([0,\infty),\mathbb{R}^d), \mathcal{C}([0,\infty),\mathbb{R}^d))$

$$\mathbb{E}^\nu[F(S) \mid \tau<\infty] = \mathbb{E}^0[F(B)]$$

and $\forall\, A \in \mathcal{F}_\tau^+$,

$$\mathbb{E}^\nu[F(S)\mathbb{1}_A \mid \tau<\infty] = \mathbb{E}^\nu[F(S) \mid \tau<\infty]\,\mathbb{P}^\nu(A \mid \tau<\infty)$$

Pf. Abusing notation slightly, denote $X_{\tau+\cdot} = X_\cdot \circ \theta_\tau$ for any process X.

Then $S_t = B_{t+\tau} - B_\tau = (B \circ \theta_\tau)_t - (B \circ \theta_\tau)_0$

$= ((B_\cdot - B_0) \circ \theta_\tau)_t$.

Hence $\mathbb{E}^\nu[F(S) \mathbb{1}_{\{\tau<\infty\}} \mid \mathcal{F}_\tau^+]$

$= \mathbb{E}^\nu[F((B_\cdot - B_0) \circ \theta_\tau) \mid \mathcal{F}_\tau^+] \mathbb{1}_{\{\tau<\infty\}}$

Strong Markov prop.

$= \mathbb{E}^x[F(B_\cdot - B_0)]\big|_{x=(B-B_0)_\tau} \mathbb{1}_{\{\tau<\infty\}}$

But $x \mapsto \mathbb{E}^x[F(B_\cdot - B_0)]$ is constant, b/c $\forall x$ $(B-B_0)_0 = 0$ $\mathbb{P}^x$-a.s.

$= \mathbb{E}^0[F(B)] \mathbb{1}_{\{\tau<\infty\}}$.

Thus, if $A \in \mathcal{F}_\tau^+$,

$$\mathbb{E}^\nu[F(S) \mathbb{1}_A \mid \tau<\infty] = \frac{1}{\mathbb{P}^\nu(\tau<\infty)} \mathbb{E}^\nu[\mathbb{1}_{\{\tau<\infty\}} F(S) \mathbb{1}_A]$$

$$\mathbb{E}^\nu[\mathbb{E}^\nu[F(S) \mathbb{1}_{\{\tau<\infty\}} \mid \mathcal{F}_\tau^+] \mathbb{1}_A]$$

$$\mathbb{E}^\nu[\mathbb{E}^0[F(B)] \mathbb{1}_{\{\tau<\infty\}} \mathbb{1}_A]$$

$$= \mathbb{E}^0[F(B)] \mathbb{E}^\nu[\mathbb{1}_A \mid \tau<\infty].$$

$\therefore\ \mathbb{E}^{\nu}[F(S)\mathbb{1}_A \mid \tau<\infty] = \mathbb{E}^{0}[F(B)]\,\mathbb{E}^{\nu}[\mathbb{1}_A \mid \tau<\infty] \quad \forall A \in \mathcal{F}_{\tau}^{+}$.

Take $A=\Omega$; shows $\mathbb{E}^{\nu}[F(S) \mid \tau<\infty] = \mathbb{E}^{0}[F(B)]\cdot 1$ ✓

$$\mathbb{E}^{\nu}[F(S)\mathbb{1}_A \mid \tau<\infty] = \mathbb{E}^{\nu}[F(S) \mid \tau<\infty]\,\mathbb{E}^{\nu}[\mathbb{1}_A \mid \tau<\infty] \quad ✓ \quad ///$$

Note: in [Lec. 55.2] we showed $B_{T+\cdot} - B_T$ is a Brownian motion indep. from $\mathcal{F}_T$ when $T \geq 0$ is constant. Now we've proved the stronger claim that it is independent from $\mathcal{F}_T^{+} \supsetneq \mathcal{F}_T$.

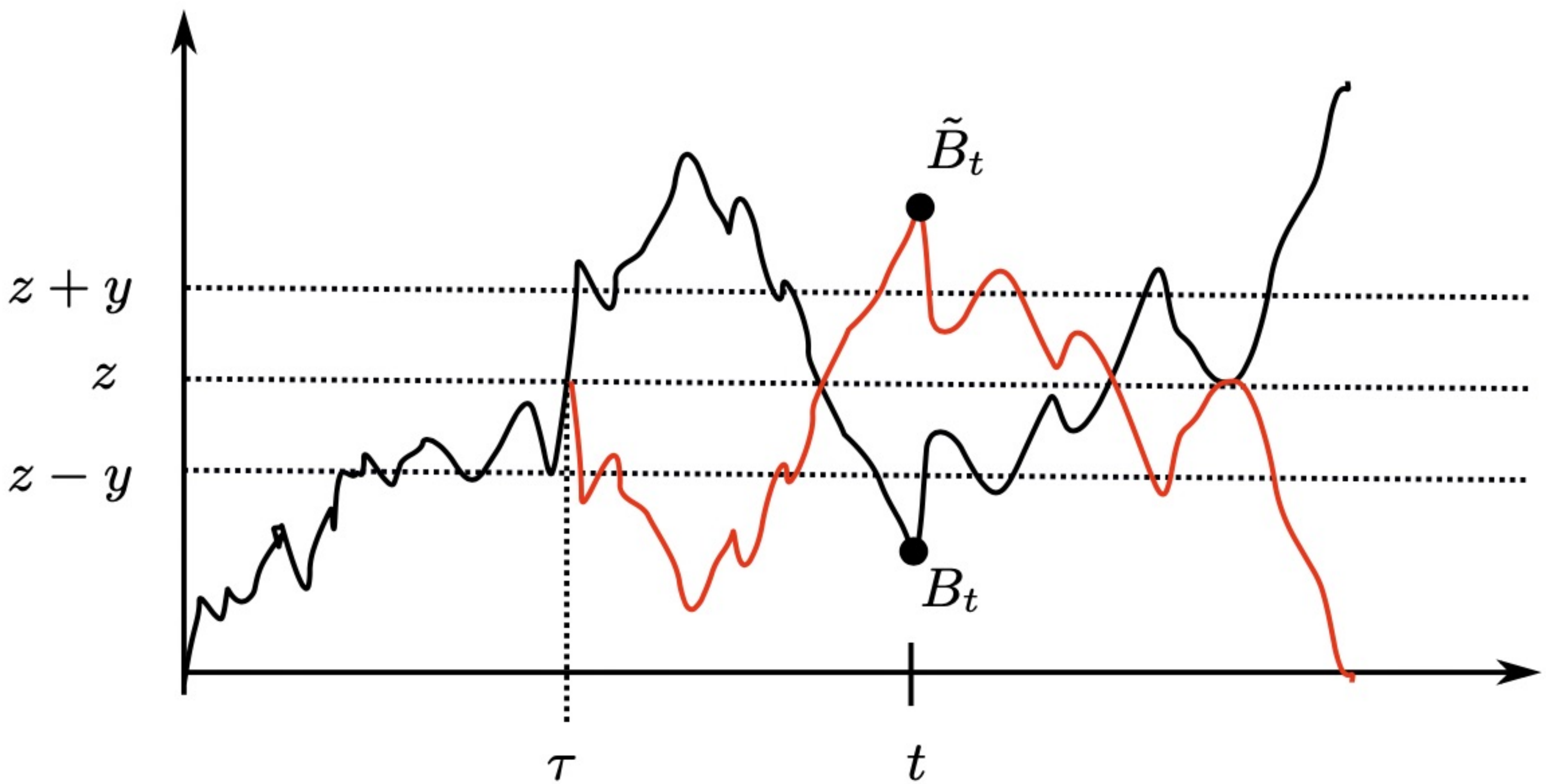
Also: if τ is an optional time, it is $\mathcal{F}_{\tau}^{+}$-measurable.
Hence: $S_t = B_{t+\tau} - B_{\tau}$ is indep. from τ.
(conditioned on $\tau < \infty$).

Theorem: (Reflection Principle)

Let $B_\cdot$ be a Brownian motion, and let τ be an optional time adapted to its natural filtration. Then

$$\tilde{B}_t := B_{t\wedge\tau} - (B_t - B_{t\wedge\tau}), \quad t \geq 0$$

is a Brownian motion.



(Reflection Principle) $\tilde{B}_t := B_{t\wedge\tau} - (B_t - B_{t\wedge\tau})$ is a Brownian motion.

Pf. It suffices to show $\tilde{B}|_{[0,T]}$ is a Brownian motion for each $T>0$.

$\therefore$ Replacing τ with $\tau\wedge T$ if needed, wlog assume $\tau<\infty$.

We $\therefore$ know that $S_t = B_{t+\tau} - B_\tau$ is a Brownian motion, indep. from $\mathcal{F}_\tau^+$.

- τ is $\mathcal{F}_\tau^+$-measurable
- $B_t^\tau = \underbrace{B_{t\wedge\tau}}$ is $\mathcal{F}_\tau^+$-measurable

[Lec. 56.3]

$\mathcal{F}_{t\wedge\tau}^+$-meas., and $t\wedge\tau \le \tau$ $\therefore$ $\mathcal{F}_{t\wedge\tau}^+ \subseteq \mathcal{F}_\tau^+$.

Thus S is independent from (τ, B^τ).

$\overset{d}{=}$
$-S$

$$(\tau, B^\tau, S) \overset{d}{=} (\tau, B^\tau, -S)$$

Now note that $S_{(t-\tau)_+} = B_{(t-\tau)_+ + \tau} - B_\tau = \begin{cases} 0, & t\le\tau \\ B_t - B_\tau, & t\ge\tau \end{cases}$

$(t-\tau)\vee 0.$

$= B_t - B_{t\wedge\tau}.$

$$B_t^\tau + S_{(t-\tau)_+} = B_t.$$

$$B_t^\tau - S_{(t-\tau)_+} = \tilde{B}_t.$$

///