Theorem: Let $(B_t)_{t\geq 0}$ be a Brownian motion on $\mathbb{R}$. Define

$$T_0 = \inf\{t>0 : B_t = 0\}$$
$$T_+ = \inf\{t>0 : B_t > 0\}$$
$$T_- = \inf\{t>0 : B_t < 0\}$$

Then $\mathbb{P}^0(T_\pm = 0) = \mathbb{P}^0(T_0 = 0) = 1$.

Pf. Since $T_\pm$ and T_0 are optional times,

$$\{T_+ = 0\}, \{T_- = 0\}, \{T_0 = 0\} \in \mathcal{F}_0^+.$$

$\therefore$ By Blumenthal, $\mathbb{P}(\quad) \in \{0,1\}$ for each.

- For any $t>0$, $\{B_t > 0\} \subseteq \{T_+ \leq t\}$

- Similar argument for T_-
- $\{T_+ = 0\} \cap \{T_- = 0\} \subseteq \{T_0 = 0\}$ by intermediate value theorem.

Cor: Let $(B_t)_{t\geq 0}$ be a Brownian motion on $\mathbb{R}$. Define

$$S_0 = \sup\{t>0 : B_t = 0\}$$
$$S_+ = \sup\{t>0 : B_t > 0\}$$
$$S_- = \sup\{t>0 : B_t < 0\}$$

Then $\mathbb{P}^0(S_\pm = \infty) = \mathbb{P}^0(S_0 = \infty) = 1$.

Pf. Let $X_t = tB_{1/t}\,\mathbb{1}_{t>0}$. Then $(X_t)_{t\geq 0}$ is a Brownian motion on $\mathbb{R}$.

$\therefore$ $T_\pm^X = T_0^X = 0$ $\mathbb{P}^0$-a.s.

So, Brownian motion oscillates wildly locally, and oscillates i.o. as $t\to\infty$.

We'd also like to understand how big/small it gets as $t \to \infty$.

The Strong Markov property gives us the tools to answer this.