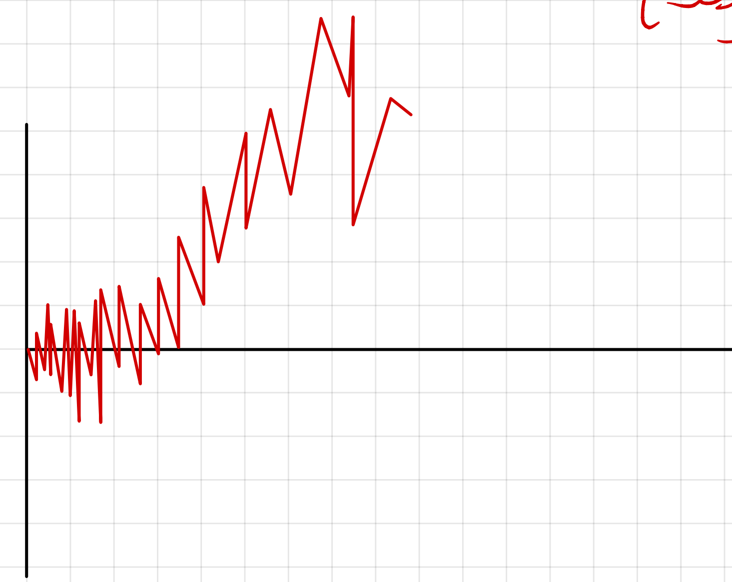
Theorem: Let $(B_t)_{t\geq 0}$ be a Brownian motion on $\mathbb{R}$. Define

$$T_0 = \inf\{t>0 : B_t = 0\}$$ Hitting time of $\{0\}$ ← closed, cont paths
$$T_+ = \inf\{t>0 : B_t > 0\}$$ Hitting times of $(0,\infty)$, $(-\infty,0)$ } open → optional
$$T_- = \inf\{t>0 : B_t < 0\}$$

Then $\mathbb{P}^0(T_\pm = 0) = \mathbb{P}^0(T_0 = 0) = 1$.



Pf. Since $T_\pm$ and T_0 are optional times,

$$\{T_+=0\}, \{T_-=0\}, \{T_0=0\} \in \mathcal{F}_0^+.$$

$\therefore$ By Blumenthal, $\mathbb{P}(\quad) \in \{0,1\}$ for each.

- For any $t>0$, $\{B_t>0\} \subseteq \{T_+ \leq t\}$

$$\frac{1}{2} = \mathbb{P}(\qquad) \leq \mathbb{P}(T_+\leq t)$$

$$\therefore \mathbb{P}^0(T_+=0) = \lim_{n\to\infty} \mathbb{P}^0(T_+ \leq \tfrac{1}{n}) \geq \frac{1}{2}$$

- Similar argument for T_- (or $T_-^B = T_+^{-B}$)
- $\{T_+=0\} \cap \{T_-=0\} \subseteq \{T_0=0\}$ by intermediate value theorem. ///

Cor: Let $(B_t)_{t\geq 0}$ be a Brownian motion on $\mathbb{R}$. Define

$$S_0 = \sup\{t>0 : B_t = 0\}$$
$$S_+ = \sup\{t>0 : B_t > 0\}$$
$$S_- = \sup\{t>0 : B_t < 0\}$$

Then $\mathbb{P}^0(S_\pm = \infty) = \mathbb{P}^0(S_0 = \infty) = 1$.

Pf. Let $X_t = tB_{1/t}\mathbb{1}_{t>0}$. Then $(X_t)_{t\geq 0}$ is a Brownian motion on $\mathbb{R}$.

$\therefore\ T_\pm^X = T_0^X = 0 \quad \mathbb{P}^0$-a.s.

$\{S_0 = \infty\} = \{B_t = 0 \text{ i.o. as } t\to\infty\}$

$= \{X_t = 0 \text{ i.o. as } t \downarrow 0\} = \{T_0^X = 0\}$

$S_\pm$ similar. ///

So, Brownian motion oscillates wildly locally, and oscillates i.o. as $t\to\infty$.

We'd also like to understand how big/small it gets as $t\to\infty$.

The Strong Markov property gives us the tools to answer this.