Let $(X_t)_{t\geq 0} : (\Omega, \mathcal{F}, \mathbb{P}) \to S$ be a time-homogeneous Markov process in a separable metric space $S$, with paths in $\Gamma = C([0,\infty), S)$ or $\Gamma = RC([0,\infty), S)$, whose transition semigroup satisfies the conditions of the Strong Markov prop.

$\exists$ multiplicative system $\mathbb{M} \subseteq C_b(S)$ s.t. $\sigma(\mathbb{M}) = \mathcal{B}(S)$ and $Q_t \mathbb{M} \subseteq \mathbb{M}$ $\forall t \geq 0$.

(E.g. Brownian motion, or any bounded rate continuous-time Markov chain.)

We will work in the natural filtration generated by the process:

$$\mathcal{F}_t = \mathcal{F}^X_t$$

Notation: Let $\Theta_t : \Gamma \to \Gamma$ denote the Markov shift operator

$$\Theta_t(\omega)(s) = \omega(t+s).$$

Note that any $t \geq 0$ is a stopping (& $\therefore$ optional) time.

$\therefore$ The Strong Markov property implies, $\forall F \in B(\Gamma, \mathcal{C}(\Gamma))$

$$\mathbb{E}[F \circ \Theta_t(X) \mid \mathcal{F}_t^+]$$

**Prop:** Let $Z \in \mathbb{B}(\Omega, \mathcal{F}_\infty)$. Then for any $t \geq 0$,

$$\mathbb{E}[Z \mid \mathcal{F}_t^+] = \mathbb{E}[Z \mid \mathcal{F}_t] \quad a.s.$$

**Pf.** First, s'pose $Z$ has the form $Z = Y \cdot F \circ \theta_t(X)$ for $F \in \mathbb{B}(\Gamma, \mathcal{C}(\Gamma))$, $Y \in \mathbb{B}(\Omega, \mathcal{F}_t)$

$$\therefore \mathbb{E}[Z \mid \mathcal{F}_t^+] = \mathbb{E}[Y \cdot F \circ \theta_t(X) \mid \mathcal{F}_t^+]$$

- $M_t = \{ Y \cdot F \circ \theta_t(X) : Y \in \mathbb{B}(\Omega, \mathcal{F}_t),\ F \in \mathbb{B}(\Gamma, \mathcal{C}(\Gamma)) \}$ is a multiplicative system

- $\mathbb{H} = \{ Z \in \mathbb{B}(\Omega, \mathcal{F}_\infty) : \mathbb{E}[Z \mid \mathcal{F}_t^+] = \mathbb{E}[Z \mid \mathcal{F}_t] \text{ a.s.} \}$
  ↳ Clearly contains $\mathbb{1}$, linear subspace
  ↳ closed under bounded convergence
- $\mathbb{M}_t \subseteq \mathbb{H}$ (proved on last slide)

$\therefore$ By Dynkin,

$$\mathbb{B}(\Omega, \sigma(\mathbb{M}_t)) \subseteq \mathbb{H}$$

Thus, we just need to show $\mathcal{F}_\infty \subseteq \sigma(\mathbb{M}_t)$.

Note: if $G \in \tilde{\mathbb{M}} = \{ \omega \mapsto f_1(\omega(t_1)) \cdots f_k(\omega(t_k)) : k \in \mathbb{N},\ 0 \leq t_1 < \cdots < t_k,\ f_1, \dots, f_k \in \mathbb{M} \}$

then $G(X) = f_1(X_{t_1}) \cdots f_{j-1}(X_{t_{j-1}}) f_j(X_{t_j}) \cdots f_k(X_{t_k})$

**Cor:** (Blumenthal's 0-1 Law)

If $(\mathcal{F}_t)_{t\geq 0}$ is the natural filtration of a "strong Markov process," then $\mathcal{F}_0^+$ is trivial: $\forall A \in \mathcal{F}_0^+$, $\mathbb{P}^x(A) \in \{0,1\}$ $\forall$ initial states $x$.

**Pf.** Take $Z = \mathbb{1}_A \in \mathcal{B}(\Omega, \mathcal{F}_0^+)$.

$$\therefore \mathbb{1}_A = \mathbb{E}[\mathbb{1}_A \mid \mathcal{F}_0^+] = \mathbb{E}[\mathbb{1}_A \mid \mathcal{F}_0]$$