Let $(X_t)_{t\geq 0} : (\Omega, \mathcal{F}, \mathbb{P}) \to S$ be a time-homogeneous Markov process in a separable metric space S, with paths in $\Gamma = C([0,\infty), S)$ or $\Gamma = RC([0,\infty), S)$, whose transition semigroup satisfies the conditions of the Strong Markov prop.

$\exists$ multiplicative system $\mathbb{M} \subseteq C_b(S)$ s.t. $\sigma(\mathbb{M}) = \mathcal{B}(S)$ and $Q_t \mathbb{M} \subseteq \mathbb{M}$ $\forall t \geq 0$.

(E.g. Brownian motion, or any bounded rate continuous-time Markov chain.)

We will work in the natural filtration generated by the process:

$$\mathcal{F}_t = \mathcal{F}^X_t = \sigma(X_s : s \leq t)$$

Notation: Let $\theta_t : \Gamma \to \Gamma$ denote the Markov shift operator

$$\theta_t(\omega)(s) = \omega(t+s).$$

Note that any $t \geq 0$ is a stopping (& $\therefore$ optional) time.

$\therefore$ The Strong Markov property implies, $\forall F \in B(\Gamma, \mathcal{C}(\Gamma))$

$$\begin{aligned} \mathbb{E}[F \circ \theta_t(X) \mid \mathcal{F}_t^+] &= \mathbb{E}[F(X_{t+\cdot}) \mid \mathcal{F}_t^+] \\ &= \mathbb{E}^x[F(X)]\big|_{x = X_t} \\ &= \mathbb{E}[F(X_{t+\cdot}) \mid \mathcal{F}_t] = \mathbb{E}[F \circ \theta_t(X) \mid \mathcal{F}_t] \end{aligned}$$

Prop: Let $Z \in \mathbb{B}(\Omega, \mathcal{F}_\infty)$. Then for any $t \geq 0$, $\mathcal{F}_t = \mathcal{F}_t^X$.

$$\mathbb{E}[Z \mid \mathcal{F}_t^+] = \mathbb{E}[Z \mid \mathcal{F}_t] \quad a.s.$$

Pf. First, s'pose Z has the form $Z = Y \cdot F \circ \theta_t(X)$ for $F \in \mathbb{B}(\Gamma, \mathcal{C}(\Gamma))$, $Y \in \mathbb{B}(\Omega, \mathcal{F}_t)$

$$\begin{aligned}
\therefore \mathbb{E}[Z \mid \mathcal{F}_t^+] &= \mathbb{E}[Y \cdot F \circ \theta_t(X) \mid \mathcal{F}_t^+] && \mathcal{F}_t \leq \mathcal{F}_t^+ \\
&= Y \cdot \mathbb{E}[F \circ \theta_t(X) \mid \mathcal{F}_t^+] && t \text{ is optional} \\
&= Y \cdot \mathbb{E}^x[F(X)]|_{x = X_t} && t \text{ is a stopping time} \\
&= Y \cdot \mathbb{E}[F \circ \theta_t(X) \mid \mathcal{F}_t] && Y \text{ is } \mathcal{F}_t\text{-meas.} \\
&= \mathbb{E}[Y \cdot F \circ \theta_t(X) \mid \mathcal{F}_t] = \mathbb{E}[Z \mid \mathcal{F}_t].
\end{aligned}$$

- $M_t = \{ Y \cdot F \circ \theta_t(X) : Y \in \mathbb{B}(\Omega, \mathcal{F}_t), F \in \mathbb{B}(\Gamma, \mathcal{C}(\Gamma)) \}$ is a multiplicative system

$$\begin{aligned}
&(Y_1 \cdot F_1 \circ \theta_t(X)) \cdot (Y_2 \cdot F_2 \circ \theta_t(X)) \\
&= \underbrace{Y_1 Y_2}_{\in \mathbb{B}(\Omega, \mathcal{F}_t)} \cdot \underbrace{(F_1 F_2)}_{\in \mathbb{B}(\Gamma, \mathcal{C}(\Gamma))} \circ \theta_t(X)
\end{aligned}$$

- $\mathbb{H} = \{ Z \in \mathbb{B}(\Omega, \mathcal{F}_\infty) : \mathbb{E}[Z | \mathcal{F}_t^+] = \mathbb{E}[Z | \mathcal{F}_t] \text{ a.s.} \}$
  ↳ Clearly contains 1, linear subspace
  ↳ closed under bounded convergence (use cDCT)
- $\mathbb{M}_t \subseteq \mathbb{H}$ (proved on last slide)

∴ By Dynkin,

$\mathbb{B}(\Omega, \sigma(\mathbb{M}_t)) \subseteq \mathbb{H}$ Thus, we just need to show $\mathcal{F}_\infty \subseteq \sigma(\mathbb{M}_t)$.

Note: if $G \in \tilde{\mathbb{M}} = \{ \omega \mapsto f_1(\omega(t_1)) \cdots f_k(\omega(t_k)) : k \in \mathbb{N},\ 0 \leq t_1 < \cdots < t_k,\ f_1, \dots, f_k \in \mathbb{M} \}$

then $G(X) = \underbrace{f_1(X_{t_1}) \cdots f_{j-1}(X_{t_{j-1}})}_{\in \mathbb{B}(\Omega, \mathcal{F}_t)} \underbrace{f_j(X_{t_j}) \cdots f_k(X_{t_k})}_{= F \circ \theta_t (X)} \in \mathbb{M}_t$. ✓

suppose $t_{j-1} < t \leq t_j$

$F(\omega) = f_j(\omega(t_j - t)) \cdots f_k(\omega(t_k - t))$

we showed $\sigma(\tilde{\mathbb{M}}) = \mathcal{C}(\Gamma) = \sigma(\pi_s : s \geq 0)$.

$\Longrightarrow \mathcal{F}_\infty = \sigma(X_s : s \geq 0) \subseteq \tilde{\mathbb{M}} \circ X \subseteq \sigma(\mathbb{M}_t)$.

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Cor (Blumenthal's 0-1 Law)

If $(\mathcal{F}_t)_{t\geq 0}$ is the natural filtration of a "strong Markov process", then $\mathcal{F}_0^+$ is trivial: $\forall A \in \mathcal{F}_0^+$, $\mathbb{P}^x(A) \in \{0,1\}$ $\forall$ initial states x.

Pf. Take $Z = \mathbb{1}_A \in B(\Omega, \mathcal{F}_0^+)$.

$\mathcal{F}_0^+ \subset \mathcal{F}_\infty$

$$\therefore \mathbb{1}_A = \mathbb{E}[\mathbb{1}_A \mid \mathcal{F}_0^+] = \underbrace{\mathbb{E}[\mathbb{1}_A \mid \mathcal{F}_0]}_{\mathcal{F}_0\text{-meas.}} = f(X_0) \quad \text{for some } f \in B(S, \mathcal{B}(S))$$

$\mathcal{F}_0 = \sigma(X_0)$

But $\mathbb{P}^x(X_0 = x) = 1$. $\therefore \mathbb{1}_A = f(x)$ a.s. $[\mathbb{P}^x]$.

$\mathbb{1}_A$: $\{0,1\}$-valued; $f(x)$: $=0$ or $=1$.

$$\therefore \mathbb{P}^x(A) = \mathbb{E}^x[\mathbb{1}_A] = \mathbb{E}^x[f(x)] = 0 \text{ or } 1.$$

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