Theorem: (Strong Markov Property) Let S be a separable metric space. Let $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered propability space, and let $(Q_t)_{t\geq 0}$ be a Markov transition semigroup of operators on $\mathbb{B}(S, \mathcal{B}(S))$.

Assume $\exists$ multiplicative system $\mathbb{M} \subseteq C_b(S)$ s.t. $\sigma(\mathbb{M}) = \mathcal{B}(S)$ and $Q_t \mathbb{M} \subseteq \mathbb{M}$ $\forall t$.

Let $(X_t)_{t\geq 0}$ be a time homogeneous Markov process with transition operators Q_t, and paths in $\Gamma = C[0,\infty)$ or $\Gamma = RC[0,\infty)$.

Then for any $F \in \mathbb{B}(\Gamma, \mathcal{C}(\Gamma))$, and any optional time $\tau : \Omega \to [0,\infty]$,

$$\mathbb{E}[F(X_{\tau+\cdot}) \mid \mathcal{F}_\tau^+] = \mathbb{E}^x[F(X_\cdot)]\big|_{x = X_\tau} \text{ a.s. on } \{\tau < \infty\}.$$

For the proof, we will make use of the already-proved special case, when $\tau(\Omega)$ is countable, and approximate τ by such countable range stopping times:

$$\tau_n = \infty \mathbb{1}_{\tau=\infty} + \sum_{k=1}^{\infty} \frac{k}{2^n} \mathbb{1}_{\frac{k-1}{2^n} \leq \tau < \frac{k}{2^n}}$$

Pf. We showed that $\tau_n \downarrow \tau$, $\mathcal{F}_\tau^+ \subseteq \mathcal{F}_{\tau_n}$, and $\{\tau_n = \infty\} = \{\tau = \infty\}$ $\forall n$.
B/c $\tau_n(\Omega)$ is countable, we've proved that, $\forall A \in \mathcal{F}_\tau^+ \subseteq \mathcal{F}_{\tau_n}$, $F \in B(\Gamma, \mathcal{C}(\Gamma))$,

$$\mathbb{E}[F(X_{\tau_n + \cdot}) \mathbb{1}_{\tau < \infty} \mathbb{1}_A]$$

Now, want to take $n \to \infty$.

Start with functions $F : \Gamma \to \mathbb{R}$ of the form $F(\omega) = f_1(\omega(t_1)) \cdots f_k(\omega(t_k))$
for some $t_k > t_{k-1} > \cdots > t_2 > t_1 \geq 0$.
and $f_1, f_2, \ldots, f_k \in M$.

Using the way a Markov process's f.d. distributions are determined by its transition operators [Lec 37.1]:

$$\mathbb{E}^x[F(X_\cdot)] =$$

Since $f_j \in M$ and $Q_t M \subseteq M$,
That is: $x \mapsto \mathbb{E}^x[F(X_\cdot)]$ is continuous.
Also, by assumption $\Gamma \subseteq RC[0,\infty)$, so since $\tau_n \downarrow \tau$,
$X_{\tau_n} \to X_\tau$ a.s.

Also, $F(X_{\tau_n+\cdot}) = \prod_{i=1}^{k} f_i(X_{\tau_n+t_i})$

And since $\|F\|_\infty \leq \|f_1\|_\infty \cdots \|f_k\|_\infty < \infty$ uniformly in n, it follows by the DCT that

$$\mathbb{E}[F(X_{\tau_n+\cdot})\mathbb{1}_{\tau<\infty}\mathbb{1}_A] \to \mathbb{E}[F(X_{\tau+\cdot})\mathbb{1}_{\tau<\infty}\mathbb{1}_A] \quad \text{as } n\to\infty$$

This proves the Strong Markov property for F of this special form.

The remainder of the proof is an application of Dynkin's mult. systems theorem.

- Let $\tilde{\mathbb{M}} = \{F \in C_b(\Gamma) : F(\omega) = \prod_{i=1}^{k} f_i(\omega(t_i))$ for some $f_i \in \mathbb{M}$ and $0 \leq t_1 < t_2 < \cdots < t_k < \infty\}$

  It is straightforward to check that $\tilde{\mathbb{M}}$ is a mult. system.
- Let $\tilde{\mathbb{H}} = \{F \in \mathbb{B}(\Gamma) : \star$ holds$\}$

  It is straightforward to check that $1 \in \tilde{\mathbb{H}}$, and $\tilde{\mathbb{H}}$ is a linear subspace closed under bdd convergence (by DCT).

We've shown that $\tilde{\mathbb{M}} \subseteq \tilde{\mathbb{H}}$. It follows by Dynkin that $\mathbb{B}(\Gamma, \sigma(\tilde{\mathbb{M}})) \subseteq \tilde{\mathbb{H}}$.

Thus, to complete the proof, we just need to show that $\mathcal{C}(\Gamma) \subseteq \sigma(\tilde{\mathbb{M}})$.

$\mathcal{C}(\Gamma) = \sigma(\pi_t : t \geq 0)$

So it suffices to show that $\pi_t : \Gamma \to S$ is $\sigma(\tilde{\mathbb{M}})$-measurable for each $t \geq 0$.

Claim: $\forall t \geq 0$ and $\forall f \in \mathbb{B}(S, \mathcal{B}(S))$, $f \circ \pi_t : \Gamma \to \mathbb{R}$ is $\sigma(\tilde{\mathbb{M}})$-measurable.

↳ With this in hand, take $f = \mathbb{1}_B$ for $B \in \mathcal{B}(S)$

To prove the claim: Dynkin again!

$\mathbb{H} = \{ f \in \mathbb{B}(S, \mathcal{B}(S)) : f \circ \pi_t$ is $\sigma(\tilde{\mathbb{M}})$-measurable$\}$

$\mathbb{M}$ = same $\mathbb{M} \subseteq C_b(S)$ from statement of theorem

↳ In particular, $\sigma(\mathbb{M}) = \mathcal{B}(S)$.

- Check easily that $1 \in \mathbb{H}$, linear subspace, closed under bdd convergence.

Thus, we have shown that: $\forall F \in B(\Gamma, \mathcal{C}(\Gamma))$ & $\forall A \in \mathcal{F}_\tau^+$,

$$\mathbb{E}[F(X_{\tau+\cdot}) \mathbb{1}_{\tau<\infty} \cdot \mathbb{1}_A] = \mathbb{E}[\mathbb{E}^x[F(X_\cdot)]|_{x=X_\tau} \mathbb{1}_{\tau<\infty} \cdot \mathbb{1}_A]$$

$\therefore$ $Z_1 := \mathbb{E}[F(X_{\tau+\cdot}) \mathbb{1}_{\tau<\infty} | \mathcal{F}_\tau^+]$

& $Z_2 := \mathbb{E}^x[F(X_\cdot)]|_{x=X_\tau} \mathbb{1}_{\tau<\infty}$

are two rv's in $\mathbb{B}(\Omega, \mathcal{F}_\tau^+)$

satisfying $\mathbb{E}[Z_1 \mathbb{1}_A] = \mathbb{E}[Z_2 \mathbb{1}_A]$ $\forall A \in \mathcal{F}_\tau^+$.