Theorem: (Strong Markov Property) Let S be a separable metric space. Let $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered propability space, and let $(Q_t)_{t\geq 0}$ be a Markov transition semigroup of operators on $\mathbb{B}(S, \mathcal{B}(S))$.

Assume $\exists$ multiplicative system $\mathbb{M} \subseteq C_b(S)$ s.t. $\sigma(\mathbb{M}) = \mathcal{B}(S)$ and $Q_t \mathbb{M} \subseteq \mathbb{M}$ $\forall t$.

Let $(X_t)_{t\geq 0}$ be a time homogeneous Markov process with transition operators Q_t, and paths in $\Gamma = C[0,\infty)$ or $\Gamma = RC[0,\infty)$.

Then for any $F \in \mathbb{B}(\Gamma, \mathcal{C}(\Gamma))$, and any optional time $\tau : \Omega \to [0,\infty]$,

$$\mathbb{E}[F(X_{\tau+\cdot}) \mid \mathcal{F}_\tau^+] = \mathbb{E}^x[F(X_\cdot)]\big|_{x = X_\tau} \text{ a.s. on } \{\tau < \infty\}.$$

For the proof, we will make use of the already-proved special case, when $\tau(\Omega)$ is countable, and approximate τ by such countable range stopping times:

$$\tau_n = \infty \mathbb{1}_{\tau=\infty} + \sum_{k=1}^{\infty} \frac{k}{2^n} \mathbb{1}_{\frac{k-1}{2^n} \leq \tau < \frac{k}{2^n}}$$

Pf. We showed that $\tau_n \downarrow \tau$, $\mathcal{F}_\tau^+ \subseteq \mathcal{F}_{\tau_n}$, and $\{\tau_n = \infty\} = \{\tau = \infty\}$ $\forall n$.

B/c $\tau_n(\Omega)$ is countable, we've proved that, $\forall A \in \mathcal{F}_\tau^+ \subseteq \mathcal{F}_{\tau_n}$, $F \in B(\Gamma, \mathcal{C}(\Gamma))$,

$$\mathbb{E}[F(X_{\tau_n+\cdot}) \mathbb{1}_{\tau < \infty} \mathbb{1}_A] = \mathbb{E}\left[\mathbb{E}^x[F(X_\cdot)]\big|_{x = X_{\tau_n}} \mathbb{1}_{\tau < \infty} \mathbb{1}_A\right].$$

Now, want to take $n \to \infty$. Need to restrict to special F.

Start with functions $F : \Gamma \to \mathbb{R}$ of the form $F(\omega) = f_1(\omega(t_1)) \cdots f_k(\omega(t_k))$

for some $t_k > t_{k-1} > \cdots > t_2 > t_1 \geq 0$.

and $f_1, f_2, \dots, f_k \in M$.

Using the way a Markov process's f.d. distributions are determined by its transition operators [Lec 37.1]: $M_f(g) = fg$.

$$\mathbb{E}^x[F(X_\cdot)] = \left(Q_{t_1} M_{f_1} Q_{t_2 - t_1} M_{f_2} \cdots M_{f_{k-1}} Q_{t_k - t_{k-1}} f_k\right)(x)$$

$\in M \in C_b(S)$

Since $f_j \in M$ and $Q_t M \subseteq M$,

That is: $x \mapsto \mathbb{E}^x[F(X_\cdot)]$ is continuous.

Also, by assumption $\Gamma \subseteq RC[0,\infty)$, so since $\tau_n \downarrow \tau$,

$X_{\tau_n} \to X_\tau$ a.s. $\therefore \mathbb{E}^x[F(X_\cdot)]\big|_{x = X_{\tau_n}} \xrightarrow{n\to\infty} \mathbb{E}^x[F(X_\cdot)]\big|_{x = X_\tau}$ a.s.

Also, $F(X_{\tau_n+\cdot}) = \prod_{i=1}^{k} f_i(X_{\tau_n+t_i}) \xrightarrow{n\to\infty} \prod_{i=1}^{k} f_i(X_{\tau+t_i}) = F(X_{\tau+\cdot})$

And since $\|F\|_\infty \leq \|f_1\|_\infty \cdots \|f_k\|_\infty < \infty$ uniformly in n, it follows by the DCT that

$$\mathbb{E}[F(X_{\tau_n+\cdot})\mathbb{1}_{\tau<\infty}\mathbb{1}_A] \to \mathbb{E}[F(X_{\tau+\cdot})\mathbb{1}_{\tau<\infty}\mathbb{1}_A] \quad \text{as} \quad n\to\infty$$

$$\| \qquad\qquad\qquad \therefore \ \| \ (\star)$$

$$\mathbb{E}[\mathbb{E}^x[F(X_\cdot)]|_{x=\tau_n}\mathbb{1}_{\tau<\infty}\mathbb{1}_A] \to \mathbb{E}[\mathbb{E}^x[F(X_\cdot)]|_{x=X_\tau}\mathbb{1}_{\tau<\infty}\mathbb{1}_A]$$

This proves the Strong Markov property for F of this special form.

The remainder of the proof is an application of Dynkin's mult. systems theorem.

- Let $\tilde{\mathbb{M}} = \{F \in C_b(\Gamma) : F(\omega) = \prod_{i=1}^{k} f_i(\omega(t_i))$ for some $f_i \in \mathbb{M}$ and $0 \leq t_1 < t_2 < \cdots < t_k < \infty\}$

  It is straightforward to check that $\tilde{\mathbb{M}}$ is a mult. system.
- Let $\tilde{\mathbb{H}} = \{F \in \mathbb{B}(\Gamma) : \star$ holds$\}$

  It is straightforward to check that $1 \in \tilde{\mathbb{H}}$, and $\tilde{\mathbb{H}}$ is a linear subspace closed under bdd convergence (by DCT).

We've shown that $\tilde{\mathbb{M}} \subseteq \tilde{\mathbb{H}}$. It follows by Dynkin that $\mathbb{B}(\Gamma, \sigma(\tilde{\mathbb{M}})) \subseteq \tilde{\mathbb{H}}$.

Thus, to complete the proof, we just need to show that $\mathcal{C}(\Gamma) \subseteq \sigma(\tilde{\mathbb{M}})$.

$\mathcal{C}(\Gamma) = \sigma(\pi_t : t \geq 0)$

So it suffices to show that $\pi_t : \Gamma \to S$ is $\sigma(\tilde{\mathbb{M}})$-measurable for each $t \geq 0$.

Claim: $\forall t \geq 0$ and $\forall f \in \mathbb{B}(S, \mathcal{B}(S))$, $f \circ \pi_t : \Gamma \to \mathbb{R}$ is $\sigma(\tilde{\mathbb{M}})$-measurable.

↳ With this in hand, take $f = \mathbb{1}_B$ for $B \in \mathcal{B}(S)$ $\therefore$ $\mathbb{1}_B \circ \pi_t$

$\therefore$ $\pi_t^{-1}(B) = \{\mathbb{1}_B \circ \pi_t = 1\}$

$\therefore \in \sigma(\tilde{\mathbb{M}})$.

$\therefore$ π_t is $\sigma(\tilde{\mathbb{M}})$-mes.

To prove the claim: Dynkin again!

$\mathbb{H} = \{ f \in \mathbb{B}(S, \mathcal{B}(S)) : f \circ \pi_t \text{ is } \sigma(\tilde{\mathbb{M}})\text{-measurable} \}$

$\mathbb{M}$ = same $\mathbb{M} \subseteq C_b(S)$ from statement of theorem

↳ In particular, $\sigma(\mathbb{M}) = \mathcal{B}(S)$.

- Check easily that $1 \in \mathbb{H}$, linear subspace, closed under bdd convergence.

For $f \in \mathbb{M}$, $f \circ \pi_t \in \tilde{\mathbb{M}}$ $\therefore$ $\mathbb{M} \subseteq \mathbb{H}$.

$\therefore$ by Dynkin, $\mathbb{B}(S, \sigma(\mathbb{M})) \subseteq \mathbb{H}$

$\overset{\shortparallel}{\mathbb{B}(S, \mathcal{B}(S))}$

Thus, we have shown that: $\forall F \in \mathcal{B}(\Gamma, \mathcal{C}(\Gamma))$ & $\forall A \in \mathcal{F}_\tau^+$,

$$\mathbb{E}[F(X_{\tau+\cdot})\mathbb{1}_{\tau<\infty} \cdot \mathbb{1}_A] = \mathbb{E}[\underbrace{\mathbb{E}^x[F(X_\cdot)]|_{x=X_\tau}\mathbb{1}_{\tau<\infty}}_{\text{fn of } X_\tau \text{, } \therefore\ \mathcal{F}_\tau^+\text{-meas. b/c } \tau \text{ is an optional time.}} \cdot \mathbb{1}_A]$$

$$\parallel$$

$$\mathbb{E}[\underbrace{\mathbb{E}[F(X_{\tau+\cdot})\mathbb{1}_{\tau<\infty} | \mathcal{F}_\tau^+]}_{\mathcal{F}_\tau^+\text{-meas.}}\mathbb{1}_A]$$

$\therefore$ $Z_1 := \mathbb{E}[F(X_{\tau+\cdot})\mathbb{1}_{\tau<\infty} | \mathcal{F}_\tau^+]$

& $Z_2 := \mathbb{E}^x[F(X_\cdot)]|_{x=X_\tau}\mathbb{1}_{\tau<\infty}$

are two rv's in $\mathcal{B}(\Omega, \mathcal{F}_\tau^+)$

satisfying $\mathbb{E}[Z_1 \mathbb{1}_A] = \mathbb{E}[Z_2 \mathbb{1}_A]$ $\forall A \in \mathcal{F}_\tau^+$.

$\Rightarrow$ $Z_1 = Z_2$ a.s.