Theorem: (Strong Markov Property) Let S be a separable metric space. Let $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered propability space, and let $(Q_t)_{t\geq 0}$ be a Markov transition semigroup of operators on $\mathbb{B}(S, \mathcal{B}(S))$.

Assume $\exists$ multiplicative system $\mathbb{M} \subseteq C_b(S)$ s.t. $\sigma(\mathbb{M}) = \mathcal{B}(S)$ and $Q_t \mathbb{M} \subseteq \mathbb{M}$ $\forall t$.

Let $(X_t)_{t\geq 0}$ be a time homogeneous Markov process with transition operators Q_t, and paths in $\Gamma = C[0,\infty)$ or $\Gamma = RC[0,\infty)$.

Then for any $F \in \mathbb{B}(\Gamma, \mathcal{C}(\Gamma))$, and any stopping time $\tau : \Omega \to [0,\infty]$,

$$\mathbb{E}[F(X_{\tau+\cdot}) \mid \mathcal{F}_\tau] = \mathbb{E}^x[F(X_\cdot)]\Big|_{x=X_\tau} \text{ a.s. on } \{\tau<\infty\}.$$

If τ is only an optional time, then

$$\mathbb{E}[F(X_{\tau+\cdot}) \mid \mathcal{F}_\tau^+] = \mathbb{E}^x[F(X_\cdot)]\Big|_{x=X_\tau} \text{ a.s. on } \{\tau<\infty\}.$$

$\mathbb{E}_{\mathcal{F}_\tau}[\quad]$

Before going into the proof, let's see how the Strong Markov Property applies to our favorite continuous time processes.

Eg. Let $(X_t)_{t\geq 0}$ be a continuous time Markov chain S countable, discrete.

Suppose the Markov semigroup is operator norm continuous: $\lim_{t\downarrow 0} \|Q_t - I\|_{op} = 0$.

(Equiv: the process has a bounded generator.)

By the Jump-Hold Description [Lec 42.1], the process has a right continuous version. This RC process has the Strong Markov Property:

$$C_b(S) = \mathbb{B}(S), \text{ and } \|Q_t f\|_\infty \leq \|f\|_\infty \quad \forall f \in \mathbb{B}(S), \forall t \text{ [Lec 34.1]}$$

$$Q_t(C_b(S)) = Q_t(\mathbb{B}(S)) \subseteq \mathbb{B}(S) = C_b(S) \quad \checkmark$$

$\therefore$ Take $\mathbb{M} = C_b(S) = \mathbb{B}(S)$

For a concrete example: Poisson processes.

$$Q_t f(x) = \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} e^{-\lambda t} f(x+n) \quad \text{[Lec 39.2]}$$

$$|Q_t f(x)| \leq \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} e^{-\lambda t} \underbrace{|f(x+n)|}_{\leq \|f\|_\infty} = \|f\|_\infty \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

$$= \|f\|_\infty.$$

E.g. Brownian motion $(B_t)_{t\geq 0}$ on $\mathbb{R}^d$.

$S = \mathbb{R}^d$, $Q_t f = f * \gamma_t$, $\gamma_t(x) = (2\pi t)^{-d/2} e^{-|x|^2/2t}$ [Lec. 38.1]

If $Z \stackrel{d}{=} N(0, I_{d\times d})$, then $\sqrt{t} Z \stackrel{d}{=} \gamma_t(x)dx$.

$\therefore f * \gamma_t(x) = \int f(x-y)\gamma_t(y)dy = \mathbb{E}[f(x+\sqrt{t}Z)]$

$|Q_t f(x)| \leq \mathbb{E}[\underbrace{|f(x+\sqrt{t}Z)|}_{\leq \|f\|_\infty}] \leq \|f\|_\infty$.

If $x_n \in \mathbb{R}^d$, $x_n \to x$, then for any $f \in C_b(\mathbb{R}^d)$,

$(Q_t f)(x_n) = \mathbb{E}[\underbrace{f(x_n+\sqrt{t}Z)}]$

$f(x_n+\sqrt{t}Z) \longrightarrow f(x+\sqrt{t}Z)$ a.s.

$|f(x_n+\sqrt{t}Z)| \leq \|f\|_\infty$

$\therefore$ by DCT $\longrightarrow \mathbb{E}[f(x+\sqrt{t}Z)] = Q_t f(x)$.

$\therefore$ Take $M = C_b(\mathbb{R}^d)$; $Q_t M \subseteq M$. Since Brownian paths are continuous, $\therefore (B_t)_{t\geq 0}$ has the Strong Markov Property.

Before proceeding with the proof of the Strong Markov Property, we remind ourselves that (at least in the path space $C[0,T]$) the cylinder σ-field is the natural one to use.

Prop: Let $T \in (0,\infty)$, and let S be a separable normed space. Let Ω_T denote the separable Banach space $C([0,T],S)$ equipped with the sup norm $\|\omega\|_\infty = \sup_{0\le t\le T} \|\omega(t)\|_S$.

Then $\mathcal{B}(\Omega_T) = \sigma(\pi_t : t \in [0,T]) = \mathcal{C}(\Omega_T)$.

$\sigma($"open" balls in $\Omega_T)$ $\qquad \pi_t : \Omega_T \to S, \ \pi_t(\omega) = \omega(t)$.

Pf. $\pi_t : \Omega_T \to S$ is Lip-1 continuous $\forall t \in [0,T]$:

$$\|\pi_t(\omega) - \pi_t(\omega')\|_S = \|\omega(t) - \omega'(t)\|_S \le \|\omega - \omega'\|_\infty$$

$\therefore$ π_t is $\mathcal{B}(\Omega_T) \to \mathcal{B}(S)$ measurable, so

$$\sigma(\pi_t : t \in [0,T]) \subseteq \mathcal{B}(\Omega_T).$$

Conversely, b/c $\omega \in \Omega_T$ is continuous,

$$\|\omega\|_\infty = \sup_{t\in\mathbb{Q}\cap[0,T]} \|\omega(t)\|_S = \sup_{t\in[0,T]\cap\mathbb{Q}} \|\pi_t(\omega)\|_S$$

$$\|\omega\|_\infty = \sup_{t\in[0,T]\cap\mathbb{Q}} \|\pi_t(\omega)\|_S$$

$\therefore$ For any point $\omega_0 \in \Omega_T$, the function

$$f_{\omega_0}(\omega) = \|\omega - \omega_0\|_\infty = \sup_{t\in[0,T]\cap\mathbb{Q}} \underbrace{\|\pi_t(\omega) - \omega_0(t)\|_S}_{\mathcal{C}([0,T],S)\text{-measurable}}.$$

is a supremum of a countable collection of cylinder-measurable functions.

$\therefore$ f_{ω_0} is $\mathcal{C}(\Omega_T) \to \mathcal{B}(\mathbb{R})$ measurable.

$\therefore$ $B_r(\omega_0) = \{\omega \in \Omega_T : f_{\omega_0}(\omega) < r\} = f_{\omega_0}^{-1}(-\infty, r) \in \mathcal{C}(\Omega_T)$.

Since Ω_T is separable, every open set is a countable union of open balls, hence in $\mathcal{C}(\Omega_T)$.

As $\mathcal{B}(\Omega_T)$ is generated by open balls,

$$\Longrightarrow \quad \mathcal{B}(\Omega_T) \subseteq \mathcal{C}(\Omega_T).$$

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