Recall the Strong Markov Property [Lec 46.1] for discrete time homogeneous Markov processes. We restate it here in greater generality.

Notation: Let S be a metric space. Let Γ denote one of the three function spaces $S^{[0,\infty)}$, $C[0,\infty)$, $RC[0,\infty)$

In each case, equip Γ with the cylinder σ-field $\mathcal{C}(\Gamma) = \sigma(\pi_t|_\Gamma : t \geq 0)$

Theorem: Let $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered propability space, and let $(q_t)_{t\geq 0}$ be a Markov transition semigroup of kernels on $S \times \mathcal{B}(S)$. Let $(X_t)_{t\geq 0}$ be a time homogeneous Markov process with paths in Γ, with transition semigroup $(q_t)_{t\geq 0}$.

Suppose $\tau : \Omega \to [0,\infty]$ is a $(\mathcal{F}_t)_{t\geq 0}$-stopping time with countable range. Then for any $F \in \mathbb{B}(\Gamma, \mathcal{C}(\Gamma))$,

$$\mathbb{E}[F(X_{\tau+\cdot}) \mid \mathcal{F}_\tau] = \mathbb{E}^x[F(X_\cdot)]\Big|_{x = X_\tau} \quad \text{a.s.}$$

on $\{\tau < \infty\}$.

Pf. Enumerate $\tau(\Omega) = \{t_n\}_{n \in N} \overset{?}{\cup} \{\infty\}$ where $N \subseteq \mathbb{N}$. Then $\{\tau < \infty\} = \bigcup_{n \in N} \{\tau = t_n\}$.

$$\mathbb{E}[\, F(X_{\tau + \cdot}) \mid \mathcal{F}_\tau]\, \mathbb{1}_{\{\tau < \infty\}}$$

(The path space structure Γ plays no role in this countable range-τ case. We include it just for the sequel.)

We're going to extend the Strong Markov property to general continuous τ (under the right conditions on the Markov process).
The approach will be to approximate any stopping time by countable range stopping times: given τ,

$$\tau_n :=$$

Lemma: Let τ be an optional time. For $n \in \mathbb{N}$, define

$$\tau_n = \frac{1}{2^n} \lceil 2^n \tau \rceil = \infty \mathbb{1}_{\tau=\infty} + \sum_{k=1}^{\infty} \frac{k}{2^n} \mathbb{1}_{\frac{k-1}{2^n} \leq \tau < \frac{k}{2^n}}$$

Then $\{\tau_n\}_{n=1}^{\infty}$ are stopping times, satisfying

1. $\tau_n \downarrow \tau$ as $n \to \infty$.
2. $\mathcal{F}_\tau^+ \subseteq \mathcal{F}_{\tau_n}$ $\forall n$.
3. $\{\tau_n = \infty\} = \{\tau = \infty\}$ $\forall n$.

Pf. To prove τ_n is a stopping time: let k be the $!$ integer w $\frac{k-1}{2^n} \leq t < \frac{k}{2^n}$.

$\{\tau_n \leq t\}$

Now, for $A \in \mathcal{F}_\tau^+$, $A \cap \{\tau_n \leq t\}$