Recall the Strong Markov Property [Lec 46.1] for discrete time homogeneous Markov processes. We restate it here in greater generality.

Notation: Let S be a metric space. Let Γ denote one of the three function spaces $S^{[0,\infty)}$, $C[0,\infty)$, $RC[0,\infty)$.
In each case, equip Γ with the cylinder σ-field $\mathcal{C}(\Gamma) = \sigma(\pi_t|_\Gamma : t \geq 0)$

$\pi_t : S^{[0,\infty)}$ $\pi_t(\omega) = \omega(t)$

Theorem: Let $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered propability space, and let $(q_t)_{t\geq 0}$ be a Markov transition semigroup of kernels on $S \times \mathcal{B}(S)$. Let $(X_t)_{t\geq 0}$ be a time homogeneous Markov process with paths in Γ, with transition semigroup $(q_t)_{t\geq 0}$.

Suppose $\tau : \Omega \to [0,\infty]$ is a $(\mathcal{F}_t)_{t\geq 0}$-stopping time with countable range. Then for any $F \in \mathbb{B}(\Gamma, \mathcal{C}(\Gamma))$,

$$\mathbb{E}[F(X_{\tau+\cdot}) \mid \mathcal{F}_\tau] = \mathbb{E}^x[F(X_\cdot)]\Big|_{x = X_\tau} \quad \text{a.s.}$$

on $\{\tau < \infty\}$.

Pf. Enumerate $\tau(\Omega) = \{t_n\}_{n\in N} \overset{?}{\cup} \{\infty\}$ where $N \subseteq \mathbb{N}$. Then $\{\tau<\infty\} = \bigcup_{n\in N} \{\tau = t_n\}$.

$$\mathbb{E}[F(X_{\tau+\cdot}) \mid \mathcal{F}_\tau] \mathbb{1}_{\{\tau<\infty\}} = \sum_{n\in N} \mathbb{E}[F(X_{\tau+\cdot}) \mid \mathcal{F}_{t_n}] \mathbb{1}_{\{\tau = t_n\}} \quad [\text{Lec } 45.3]$$

$\in \mathcal{F}_{t_n}$

$$= \sum_{n\in N} \mathbb{E}[F(X_{t_n+\cdot}) \mathbb{1}_{\{\tau=t_n\}} \mid \mathcal{F}_{t_n}]$$

$$= \sum_{n\in N} \underbrace{\mathbb{E}[F(X_{t_n+\cdot}) \mid \mathcal{F}_{t_n}]}_{\mathbb{E}^x[F(X_\cdot)]\big|_{x=X_{t_n}}} \mathbb{1}_{\{\tau=t_n\}}$$

(usual Markov prop.)

$$= \mathbb{E}^x[F(X)]\big|_{x=\tau} \cdot \mathbb{1}_{\{\tau<\infty\}} \quad ///$$

(The path space structure Γ plays no role in this countable range-τ case. We include it just for the sequel.)

We're going to extend the Strong Markov property to general continuous τ (under the right conditions on the Markov process).

The approach will be to approximate any stopping time by countable range stopping times: given τ,

$$\tau_n := \frac{1}{2^n}\lceil 2^n \tau \rceil = \infty \mathbb{1}_{\tau=\infty} + \sum_{k=1}^{\infty} \frac{k}{2^n} \mathbb{1}_{\frac{k-1}{2^n} < \tau < \frac{k}{2^n}}.$$

Lemma: Let τ be an optional time. For $n \in \mathbb{N}$, define

$$\tau_n = \frac{1}{2^n}\lceil 2^n \tau \rceil = \infty \mathbb{1}_{\tau=\infty} + \sum_{k=1}^{\infty} \frac{k}{2^n} \mathbb{1}_{\frac{k-1}{2^n} \leq \tau < \frac{k}{2^n}}$$

Then $\{\tau_n\}_{n=1}^{\infty}$ are stopping times, satisfying

1. $\tau_n \downarrow \tau$ as $n \to \infty$.
2. $\mathcal{F}_\tau^+ \subseteq \mathcal{F}_{\tau_n}$ $\forall n$.
3. $\{\tau_n = \infty\} = \{\tau = \infty\}$ $\forall n$. from def^n.

Pf. To prove τ_n is a stopping time: let k be the ! integer w/ $\frac{k-1}{2^n} \leq t < \frac{k}{2^n}$.

$$\{\tau_n \leq t\} = \{\tau_n \leq \tfrac{k-1}{2^n}\} = \bigcup_{j=1}^{k-1} \underbrace{\{\tau_n = \tfrac{j}{2^n}\}}_{= \{\frac{j-1}{2^n} \leq \tau < \frac{j}{2^n}\}} = \{\tau < \tfrac{k-1}{2^n}\} \in \mathcal{F}_{\frac{k-1}{2^n}} \subseteq \mathcal{F}_t.$$

Now, for $A \in \mathcal{F}_\tau^+$, $A \cap \{\tau_n \leq t\}$

$$= A \cap \{\tau < \tfrac{k-1}{2^n}\} \in \mathcal{F}_{\frac{k-1}{2^n}} \subseteq \mathcal{F}_t.$$

$$\Rightarrow A \in \mathcal{F}_{\tau_n}.$$

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