Progressive measurability allows evaluating a process $X.$ at a random time $\tau$ (and preserving measurability). What can we say about $\sigma(X_\tau)$?

**Def**: Let $\tau : (\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}) \to [0, \infty]$ be $\mathcal{F}_\infty$-measurable. Define

$$\mathcal{F}_\tau := \{ A \in \mathcal{F}_\infty : \{\tau \leq t\} \cap A \in \mathcal{F}_t \ \forall t \in [0, \infty] \}$$

and $\mathcal{F}_\tau^+ := \{ A \in \mathcal{F}_\infty : \{\tau < t\} \cap A \in \mathcal{F}_t \ \forall t \in [0, \infty] \}$

These are both sub-$\sigma$-fields of $\mathcal{F}_\infty$; the proof is essentially identical to [Lec 45.2].

(Might be tempted to denote $\mathcal{F}_\tau^+$ the set $\{ A \in \mathcal{F}_\infty : \{\tau \leq t\} \cap A \in \mathcal{F}_t^+ \ \forall t \in [0,\infty] \}$. In fact, we can: if $\tau$ is an optional time, they're equal. [HW] )

**Lemma**: If $\tau$ is a stopping time, $\tau$ is $\mathcal{F}_\tau$-measurable.

**Pf**. For $s, t \in [0, \infty)$, $\{\tau \leq s\} \cap \{\tau \leq t\}$

**Prop:** Let $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a filtered probability space.

Let $\tau$ be a stopping time, and let $(X_t)_{t \geq 0}$ be a progressively measurable process.

Then $X_\tau$ is $(\mathcal{F}_\tau)_{\{\tau < \infty\}}$-measurable.

**Pf.** For $T \in [0, \infty)$, define two maps: $\{\tau \leq T\} \xrightarrow{\Psi_T} [0,T] \times \Omega \xrightarrow{\varphi^T} S$

$$\Psi_T(\omega) = \qquad\qquad\qquad \varphi^T(t, \omega)$$

If $A \in \mathcal{F}_T$ and $a \in [0, T]$,

$$\Psi_T^{-1}([0, a] \times A) =$$

$\therefore \Psi_T$ is $(\mathcal{F}_T)_{\{\tau \leq T\}} \to \mathcal{B}[0,T] \otimes \mathcal{F}_T$ measurable.

$\therefore \varphi^T \circ \Psi_T$ is $(\mathcal{F}_T)_{\{\tau \leq T\}} \to \mathcal{B}$ measurable

Now, for $V \in \mathcal{B}$,

$$X_\tau^{-1}(V) \cap \{\tau \leq T\}$$

Thus $X_\tau^{-1}(V) \in \mathcal{F}_\tau$. And $X_\tau^{-1}(V) \subseteq \{\tau < \infty\}$ by def.

**Prop**: (Basic properties of stopping times)

1. If $\sigma, \tau$ are stopping times, so is $\sigma + \tau$. (If they are only optional times, but $\sigma > 0, \tau > 0$ a.s., then $\sigma + \tau$ is a stopping time.)

2. If $\sigma, \tau$ are stopping times, so are $\sigma \wedge \tau$ and $\sigma \vee \tau$.

3. If $\{\tau_n\}_{n=1}^{\infty}$ are optional times, then so are $\sup_n \tau_n$, $\inf_n \tau_n$, $\limsup\limits_{n \to \infty} \tau_n$, $\liminf\limits_{n \to \infty} \tau_n$.

   ↳ If they are stopping times, so is $\sup_n \tau_n$.

**Prop**: (Basic properties of stopped $\sigma$-fields)

Let $\sigma, \tau$ be stopping times.

1. $(\mathcal{F}_\tau)_{\{\tau = t\}} = (\mathcal{F}_t)_{\{\tau = t\}}$

2. If $\sigma \leq \tau$ then $\mathcal{F}_\sigma \subseteq \mathcal{F}_\tau$.

3. $(\mathcal{F}_\sigma)_{\{\sigma \leq \tau\}} \subseteq \mathcal{F}_{\sigma \wedge \tau}$   (so $\{\sigma \leq \tau\}$, $\{\sigma < \tau\}$ are in $\mathcal{F}_{\sigma \wedge \tau}$.)

4. $\mathcal{F}_\sigma \cap \mathcal{F}_\tau = \mathcal{F}_{\sigma \wedge \tau}$

Moreover, all these hold for optional times $\sigma, \tau$ if

we replace $\mathcal{F} \to \mathcal{F}^+$ everywhere.

(Proofs very similar to [Lec 45]; see [Driver, §34.2]