Progressive measurability allows evaluating a process $X_\cdot$ at a random time τ (and preserving measurability). What can we say about $\sigma(X_\tau)$?

Def: Let $\tau : (\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}) \to [0,\infty]$ be $\mathcal{F}_\infty$-measurable. Define

$$\mathcal{F}_\tau := \{ A \in \mathcal{F}_\infty : \{\tau \leq t\} \cap A \in \mathcal{F}_t \ \forall t \in [0,\infty]\}$$

$$\text{and} \quad \mathcal{F}_\tau^+ := \{ A \in \mathcal{F}_\infty : \{\tau < t\} \cap A \in \mathcal{F}_t \ \forall t \in [0,\infty]\}$$

($\mathcal{F}_\tau \subseteq \mathcal{F}_\tau^+$)

These are both sub-σ-fields of $\mathcal{F}_\infty$; the proof is essentially identical to [Lec 4.5.2].

(Might be tempted to denote $\mathcal{F}_\tau^+$ the set $\{ A \in \mathcal{F}_\infty : \{\tau \leq t\} \cap A \in \mathcal{F}_t^+ \ \forall t \in [0,\infty]\}$.
In fact, we can: if τ is an optional time, they're equal. [HW])

Lemma: If τ is a ~~stopping~~ optional time, τ is $\mathcal{F}_\tau$ ($\mathcal{F}_\tau^+$)-measurable.

Pf. For $s, t \in [0,\infty)$, $\{\tau \leq s\} \cap \{\tau \leq t\} = \{\tau \leq s \wedge t\} \in \mathcal{F}_{s\wedge t} \subseteq \mathcal{F}_t$

$\{\tau \leq s\} \in \mathcal{F}_s \subseteq \mathcal{F}_\infty$ $\quad \therefore \forall s, \ \{\tau \leq s\} \cap \{\tau \leq t\} \in \mathcal{F}_t$

$\therefore A \in \mathcal{F}_\tau$.

$\therefore \forall s \geq 0, \ \{\tau \leq s\} \in \mathcal{F}_\tau$

$\Rightarrow \tau$ is $\mathcal{F}_\tau / \mathcal{B}[0,\infty]$-meas. ///

Prop. Let $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space. (valued in $(S, \mathcal{B})$)
Let τ be a stopping time, and let $(X_t)_{t\geq 0}$ be a progressively measurable process.
Then X_τ is $(\mathcal{F}_\tau)_{\{\tau<\infty\}}$-measurable. $(\mathcal{F})_B = \{A\cap B : A\in\mathcal{F}\}$

Pf. For $T\in[0,\infty)$, define two maps: $\{\tau\leq T\} \xrightarrow{\psi_T} [0,T]\times\Omega \xrightarrow{\varphi^T} S$

$\psi_T(\omega) = (\tau(\omega), \omega)$ $\qquad \varphi^T(t,\omega) = X_t(\omega)$

$\mathcal{B}[0,T]\otimes\mathcal{F}_T \to \mathcal{B}$-meas.

If $A\in\mathcal{F}_T$ and $a\in[0,T]$,

$$\psi_T^{-1}([0,a]\times A) = \{\tau\leq a\}\cap A \subseteq \{\tau\leq T\}\cap A \in (\mathcal{F}_T)_{\{\tau\leq T\}}$$

$\therefore$ ψ_T is $(\mathcal{F}_T)_{\{\tau\leq T\}} \to \mathcal{B}[0,T]\otimes\mathcal{F}_T$ measurable.

$\therefore$ $\varphi^T\circ\psi_T$ is $(\mathcal{F}_T)_{\{\tau\leq T\}} \to \mathcal{B}$ measurable

Now, for $V\in\mathcal{B}$,

$$\begin{aligned} X_\tau^{-1}(V)\cap\{\tau\leq T\} &= \{\omega\in\Omega : \tau(\omega)\leq T \ \&\ X_{\tau(\omega)}(\omega)\in V\} \\ &= \{\omega\in\Omega : \tau(\omega)\leq T \ \&\ \varphi^T\circ\psi_T(\omega)\in V\} \\ &= \{\tau\leq T\}\cap(\varphi^T\circ\psi_T)^{-1}(V) \in \mathcal{F}_T. \end{aligned}$$

Thus $X_\tau^{-1}(V)\in\mathcal{F}_\tau$. And $X_\tau^{-1}(V)\subseteq\{\tau<\infty\}$ by def. ///

Prop: (Basic properties of stopping times)

1. If σ, τ are stopping times, so is $\sigma + \tau$. (If they are only optional times, but $\sigma > 0, \tau > 0$ a.s., then $\sigma + \tau$ is a stopping time.)
2. If σ, τ are stopping times, so are $\sigma \wedge \tau$ and $\sigma \vee \tau$.
3. If $\{\tau_n\}_{n=1}^{\infty}$ are optional times, then so are $\sup_n \tau_n$, $\inf_n \tau_n$, $\limsup_{n\to\infty} \tau_n$, $\liminf_{n\to\infty} \tau_n$.
   ↳ If they are stopping times, so is $\sup_n \tau_n$.

Prop: (Basic properties of stopped σ-fields)

Let σ, τ be stopping times.

1. $(\mathcal{F}_\tau)_{\{\tau = t\}} = (\mathcal{F}_t)_{\{\tau = t\}}$
2. If $\sigma \leq \tau$ then $\mathcal{F}_\sigma \subseteq \mathcal{F}_\tau$.
3. $(\mathcal{F}_\sigma)_{\{\sigma \leq \tau\}} \subseteq \mathcal{F}_{\sigma \wedge \tau}$ (so $\{\sigma \leq \tau\}, \{\sigma < \tau\}$ are in $\mathcal{F}_{\sigma \wedge \tau}$.)
4. $\mathcal{F}_\sigma \cap \mathcal{F}_\tau = \mathcal{F}_{\sigma \wedge \tau}$

Moreover, all these hold for optional times σ, τ if we replace $\mathcal{F} \to \mathcal{F}^+$ everywhere.

(Proofs very similar to [Lec 45]; see [Driver, §34.2])