Def: Given $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, a stopping time $\tau : \Omega \to [0,\infty]$ is a random variable satisfying $\{\tau \leq t\} \in \mathcal{F}_t \quad \forall t \geq 0$.

Note: in the discrete time setting $\{\tau < t\}$
Here we have a richer structure.

τ is called an optional time if $\{\tau < t\} \in \mathcal{F}_t \quad \forall t > 0$.

↳ Stopping times are optional times:
↳ The converse is generally false. But:

Def: For $0 \leq t < \infty$, $\mathcal{F}_t^+ :=$

For $0 < t \leq \infty$, $\mathcal{F}_t^- :=$

Each of $(\mathcal{F}_t^+)_{t\geq 0}$ and $(\mathcal{F}_t^-)_{t\geq 0}$ are filtrations,

$$\mathcal{F}_t^- \subseteq \mathcal{F}_t \subseteq \mathcal{F}_t^+ \quad \forall t$$

A filtration is right-continuous if $\mathcal{F}_t^+ = \mathcal{F}_t \quad \forall t \geq 0$.

Notice : $(\mathcal{F}_t^+)_{t\geq 0}$ is right-continuous

So every filtration has a right continuous extension.

Lemma: $\tau : \Omega \to [0,\infty]$ is a $(\mathcal{F}_t)_{t\geq 0}$-optional time iff it is a $(\mathcal{F}_t^+)_{t\geq 0}$-stopping time.

Ergo: if the filtration is right-continuous, {optional times} = { stopping times }.

Pf. If τ is a $(\mathcal{F}_t)_{t\geq 0}$-optional time, for $t\geq 0$, $\{\tau < t+\frac{1}{n}\} \downarrow \{\tau \leq t\}$.

$$\therefore \{\tau \leq t\} = \bigcap_{n\in\mathbb{N}} \{\tau < t+\tfrac{1}{n}\}$$

Conversely, if τ is a $(\mathcal{F}_t^+)_{t\geq 0}$-stopping time, for $t>0$,

$$\{\tau < t\} = \bigcup_{n\in\mathbb{N}} \{\tau \leq t-\tfrac{1}{n}\}$$

If $t=0$, $\{\tau < 0\} = \phi \in \mathcal{F}_0$.

It is $\therefore$ customary to extend the filtration, and always assume it is right-continuous.

The canonical examples of stopping times in discrete time are hitting times of a stochastic process in some set $A \subseteq S$.

Def: Let $(X_t)_{t \geq 0} : (\Omega, \mathcal{F}, \mathbb{P}) \to (S, \mathcal{B})$ be a stochastic process. Let $A \in \mathcal{B}$.

Hitting Time: $T_A := \inf\{t > 0 : X_t \in A\}$

Debut Time: $D_A := \inf\{t \geq 0 : X_t \in A\}$

Note that $D_A = T_A$ on $\{X_0 \notin A\}$; generally, $D_A \leq T_A$.

These are not necessarily stopping/optional times.

Eg. Let $E \in \mathcal{F}$ be an event with $\mathbb{P}(E) \notin \{0, 1\}$. Define

$$X_t = \max\{0, t-1\} \mathbb{1}_E + \max\{0, t-2\} \mathbb{1}_{E^c}.$$

Set $A = (0, \infty)$.

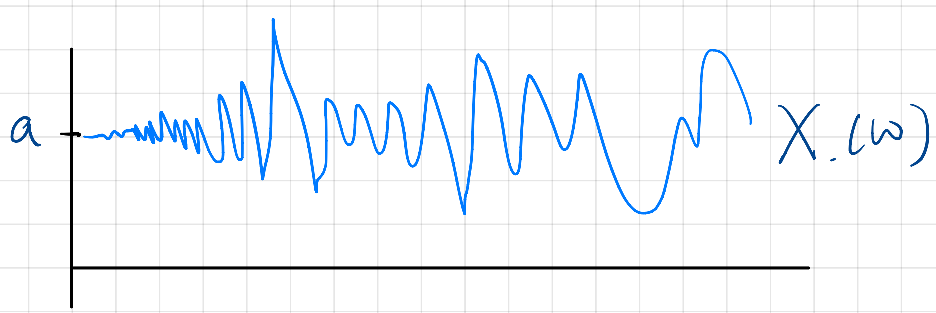
Prop: Let $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space, and let $(X_t)_{t\geq 0} : \Omega \to (S, \mathcal{B})$ be an adapted process, with right-continuous paths. Then

$\therefore (X_t)_{t\geq 0}$ is progressively measurable. So X_τ is measurable $\forall$ rv τ.

1. If $A \subseteq S$ is open, $T_A = D_A$ is an optional time.
2. If $A \subseteq S$ is closed, then on $\{T_A < \infty\}$, $X_{T_A} \in A$; on $\{D_A < \infty\}$, $X_{D_A} \in A$.

Moreover, if $X.$ has continuous paths,

3. If $A \subseteq S$ is closed, then D_A is a stopping time.
4. If $A \subseteq S$ is closed, then T_A is an optional time – and almost a stopping time:



Pf. 1. First, $T_A = D_A$ on $\{X_0 \notin A\}$ always. On $\{X_0 \in A\}$,

1. $\{D_A < t\} = \{\exists s \in [0,t) \ X_s \in A\}$

2. If A is closed, and $T_A(\omega) < \infty$, $T_A(\omega) = \inf\{t > 0 : X_t \in A\}$

3. If A is closed and $X_\cdot$ is continuous,

$\{D_A > t\} = \{X_s \notin A \ \forall s \leq t\}$

Since $X_\cdot$ is continuous, $X_{[0,t]}(\omega) := \{X_s(\omega) : 0 \leq s \leq t\}$ is compact.

$\therefore \ d(X_{[0,t]}(\omega), A)$

4. [If $A \subseteq S$ is closed and $X_\cdot$ is continuous, then $\{T_A \leq t\} \in \mathcal{F}_t \ \forall t > 0$, $\{T_A = 0\} \in \mathcal{F}_0^+$.]

- $(t > 0)$ $\quad T_A > t \iff \{X_s\}_{0 < s \leq t} \cap A = \emptyset$

- $(t = 0)$ $\quad T_A > 0 \iff \exists \delta > 0 \ X_s \in A^c \ \forall s \in (0, \delta)$