Def: Given $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, a stopping time $\tau: \Omega \to [0,\infty]$ is a random variable satisfying $\{\tau \leq t\} \in \mathcal{F}_t \quad \forall t \geq 0$.

iff $\{\tau > t\} \in \mathcal{F}_t$ ~~iff~~ $\{\tau = t\} \in \mathcal{F}_t$

Note: in the discrete time setting $\{\tau < t\} \not{=} \{\tau \leq t-1\} \in \mathcal{F}_{t-1}$

Here we have a richer structure.

τ is called an optional time if $\{\tau < t\} \in \mathcal{F}_t \quad \forall t > 0$.

↳ Stopping times are optional times: $\{\tau < t\} = \bigcup_{n\in\mathbb{N}} \underbrace{\{\tau \leq t - \tfrac{1}{n}\}}_{\in \mathcal{F}_{t-\frac{1}{n}} \subseteq \mathcal{F}_t} \in \mathcal{F}_t$.

↳ The converse is generally false. But:

Def: For $0 \leq t < \infty$, $\mathcal{F}_t^+ := \bigcap_{s>t} \mathcal{F}_s$ $\qquad \mathcal{F}_\infty^+ := \mathcal{F}_\infty^- = \mathcal{F}_\infty$.

For $0 < t \leq \infty$, $\mathcal{F}_t^- := \sigma\left(\bigcup_{s<t} \mathcal{F}_s\right)$ $\qquad \mathcal{F}_0^- := \mathcal{F}_0$

Each of $(\mathcal{F}_t^+)_{t\geq 0}$ and $(\mathcal{F}_t^-)_{t\geq 0}$ are filtrations,

$$\mathcal{F}_t^- \subseteq \mathcal{F}_t \subseteq \mathcal{F}_t^+ \qquad \forall t$$

A filtration is right-continuous if $\mathcal{F}_t^+ = \mathcal{F}_t \quad \forall t \geq 0$.

Notice: $(\mathcal{F}_t^+)_{t\geq 0}$ is right-continuous $\mathcal{F}_t^{++} = \bigcap_{s>t} \mathcal{F}_s^+ = \bigcap_{s>t} \bigcap_{r>s} \mathcal{F}_r = \bigcap_{r>t} \mathcal{F}_r = \mathcal{F}_t^+$

So every filtration has a right continuous extension.

Lemma: $\tau : \Omega \to [0,\infty]$ is a $(\mathcal{F}_t)_{t\geq 0}$-optional time ✓

iff it is a $(\mathcal{F}_t^+)_{t\geq 0}$-stopping time. ✓

Ergo: if the filtration is right-continuous, {optional times} = { stopping times }.

Pf. If τ is a $(\mathcal{F}_t)_{t\geq 0}$-optional time, for $t\geq 0$, $\{\tau < t+\frac{1}{n}\} \downarrow \{\tau \leq t\}$.

$$\therefore \{\tau \leq t\} = \bigcap_{n\in\mathbb{N}} \{\tau < t+\tfrac{1}{n}\} \in \mathcal{F}_t^+ .$$

Conversely, if τ is a $(\mathcal{F}_t^+)_{t\geq 0}$-stopping time, for $t>0$,

$$\{\tau < t\} = \bigcup_{n\in\mathbb{N}} \{\tau \leq t-\tfrac{1}{n}\} \in \mathcal{F}_t^- \subseteq \mathcal{F}_t .$$

If $t=0$, $\{\tau<0\} = \emptyset \in \mathcal{F}_0$.

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It is $\therefore$ customary to extend the filtration, and always assume it is right-continuous.

The canonical examples of stopping times in discrete time are hitting times of a stochastic process in some set $A \subseteq S$.

Def: Let $(X_t)_{t\geq 0} : (\Omega, \mathcal{F}, \mathbb{P}) \to (S, \mathcal{B})$ be a stochastic process. Let $A \in \mathcal{B}$.

Hitting Time: $\quad T_A := \inf\{t > 0 : X_t \in A\} \qquad (\inf \emptyset = \infty.)$

Debut Time: $\quad D_A := \inf\{t \geq 0 : X_t \in A\}$

Note that $D_A = T_A$ on $\{X_0 \notin A\}$; generally, $D_A \leq T_A$.

These are <u>not</u> necessarily stopping / optional times.

Eg. Let $E \in \mathcal{F}$ be an event with $P(E) \notin \{0,1\}$. Define

$$X_t = \max\{0, t-1\}\mathbb{1}_E + \max\{0, t-2\}\mathbb{1}_{E^c}.$$

Set $A = (0, \infty)$. $\quad T_A = D_A = \begin{cases} 1 & \text{on } E \\ 2 & \text{on } E^c \end{cases} \qquad \therefore \{T_A \leq 1\} = E.$

But $X_t = 0 \;\; \forall t \in [0,1] \;\; \therefore \;\; \mathcal{F}_t^X = \{\emptyset, \Omega\} \;\; \forall t \leq 1.$

$$\{T_A \leq 1\} \notin \mathcal{F}_1^X \not\ni E.$$

Prop: Let $(\Omega, (\mathcal{F}_t)_{t\ge 0}, P)$ be a filtered probability space, and let $(X_t)_{t\ge 0} : \Omega \to (S, \mathcal{B})$ be an adapted process, with right-continuous paths. Then

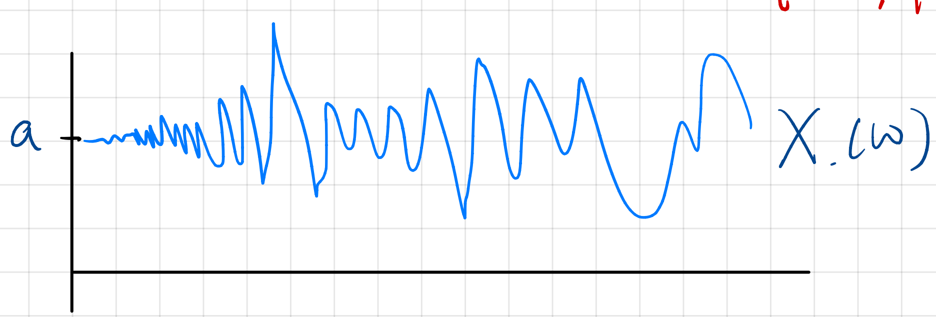
$\therefore (X_t)_{t\ge 0}$ is progressively measurable. So X_τ is measurable $\forall$ rv τ.

1. If $A \subseteq S$ is open, $T_A = D_A$ is an optional time.
2. If $A \subseteq S$ is closed, then on $\{T_A < \infty\}$, $X_{T_A} \in A$; on $\{D_A < \infty\}$, $X_{D_A} \in A$.

Moreover, if $X_\cdot$ has continuous paths,

3. If $A \subseteq S$ is closed, then D_A is a stopping time.
4. If $A \subseteq S$ is closed, then T_A is an optional time – and almost a stopping time:

$\{T_A \le t\} \in \mathcal{F}_t \ \forall t > 0$; $\{T_A = 0\} \in \mathcal{F}_0^+$



$\leftarrow$ $T_A(\omega) = 0$.

Impossible to tell from $\sigma(X_0)$

$\therefore \{T_A = 0\} \notin \mathcal{F}_0^X$.

Pf. 1. First, $T_A = D_A$ on $\{X_0 \notin A\}$ always. On $\{X_0 \in A\}$,

$\lim_{t \downarrow 0} X_t = X_0 \in A$

$\therefore$ $X_t \in A$ $\forall$ suff small $t > 0$. $\therefore T_A = 0 = D_A$.

1. $\{D_A < t\} = \{\exists s \in [0,t) \;\; X_s \in A\} = \{\exists s \in [0,t) \cap \mathbb{Q} \;\; X_s \in A\}$

$= \bigcup_{s \in [0,t) \cap \mathbb{Q}} \underbrace{\{X_s \in A\}}_{\in \mathcal{F}_s \subseteq \mathcal{F}_t^- \subseteq \mathcal{F}_t} \in \mathcal{F}_t$ ✓

$X_s = \lim_{r \downarrow s} X_r \in A \Rightarrow X_r \in A \;\; \forall r \in (s, s+\varepsilon)$

2. If A is closed, and $T_A(\omega) < \infty$, $\quad T_A(\omega) = \inf\{t > 0 : X_t \in A\}$

Similar for D_A ✓

$\exists\, t_n \downarrow T_A$ s.t. $X_{t_n} \in A$

$\therefore X_{T_A} = \lim_{n \to \infty} X_{t_n}$

$\therefore \in A$.

3. If A is closed and $X.$ is continuous,

$\{D_A > t\} = \{X_s \notin A \;\; \forall s \le t\}$

$\subseteq$ always $\quad \supseteq$ by r.c. and openness A^c

Since $X.$ is continuous, $X_{[0,t]}(\omega) := \{X_s(\omega) : 0 \le s \le t\}$ is compact.

$\therefore d(X_{[0,t]}(\omega), A) > 0 \quad (> \tfrac{1}{n(\omega)})$.

$\{D_A > t\} = \bigcup_{n \in \mathbb{N}} \{d(X_{[0,t]}, A) > \tfrac{1}{n}\}$

$= \bigcup_{n \in \mathbb{N}} \bigcap_{s \in [0,t] \cap \mathbb{Q}} \underbrace{\{d(X_s, A) > \tfrac{1}{n}\}}_{\in \mathcal{F}_s \subseteq \mathcal{F}_t} \in \mathcal{F}_t$ ✓

4. [If $A \subseteq S$ is closed and X is continuous, then $\{T_A \le t\} \in \mathcal{F}_t \ \forall t > 0$, $\{T_A = 0\} \in \mathcal{F}_0^+$.]

$\{T_A < t\} = \bigcup_{s \in \mathbb{Q} \cap (0,t)} \{T_A \le s\} \in \mathcal{F}_s \subseteq \mathcal{F}_t$

$\therefore T_A$ is an optional time.

• $(t > 0)$ $\quad T_A > t \iff \{X_s\}_{0 < s \le t} \cap A = \emptyset$

$\iff \{X_s\}_{\delta \le s \le t} \cap A = \emptyset \quad \forall \delta \in (0,t)$.

$\iff d(\{X_s\}_{\delta \le s \le t}, A) > 0$.

$$\{T_A > t\} \subseteq \bigcap_{\substack{n \in \mathbb{N} \\ n > 1/t}} \bigcup_{m \in \mathbb{N}} \Big\{ \underbrace{d(\{X_s\}_{s \in [\frac{1}{n}, t]}, A) \ge \tfrac{1}{m}}_{\bigcap_{s \in [\frac{1}{n}, t] \cap \mathbb{Q}} \{d(X_s, A) \ge \frac{1}{m}\} \in \mathcal{F}_s} \Big\} \in \mathcal{F}_t.$$

• $(t = 0)$ $\quad T_A > 0 \iff \exists \delta > 0 \ X_s \in A^c \ \forall s \in (0, \delta)$

$$\underbrace{\{T_A > 0\} \subseteq \bigcup_{n \in \mathbb{N}} \bigcap_{s \in (0, \frac{1}{n}) \cap \mathbb{Q}} \{\underbrace{X_s \in A^c}_{\in \mathcal{F}_s}\}}_{\in \mathcal{F}_0^+} \in \mathcal{F}_0^+.$$

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