Progressive Measurability

If $(X_n)_{n\in\mathbb{N}} : (\Omega, \mathcal{F}, \mathbb{P}) \to (S, \mathcal{B})$ is a discrete-time stochastic process, and $\tau : (\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{N}$ is a (measurable) function, then

$$X_\tau(\omega) = X_{\tau(\omega)}(\omega) \quad \text{is measurable.}$$

Indeed, $\varphi(n,\omega) = X_n(\omega)$ is $2^{\mathbb{N}} \otimes \mathcal{F} \to \mathcal{B}$ measurable $\quad X_\tau = \varphi \circ (\tau \times Id)$

This automatic measurability doesn't always hold in continuous time.

E.g. $(X_t)_{t\geq 0} : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$

$$X_t(\omega) = \mathbb{1}_A(t) \quad \forall \omega \in \Omega \qquad A \subset [0,\infty),\ A \notin \mathcal{B}[0,\infty)$$

then X_t is measurable for each t

but

$\varphi(t,\omega) = X_t(\omega) = \mathbb{1}_A(t)$ is not $\mathcal{B}[0,\infty) \otimes \mathcal{F} \to \mathcal{B}(\mathbb{R})$ measurable.

$$\{(t,\omega) : \varphi(t,\omega) = 1\} = A \times \Omega$$

Things get even hairier if a filtration is around.

Def: Given a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathcal{F}, \mathbb{P})$, a stochastic process $(X_t)_{t\geq 0} : (\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}) \to (S, \mathcal{B})$ is **progressively measurable** if $\forall T \geq 0$, the map $\varphi^T : [0,T] \times \Omega \to S$

$$\varphi^T(t, \omega) = X_t(\omega)$$

is $\mathcal{B}[0,T] \otimes \mathcal{F}_T \to \mathcal{B}$ measurable.

This is stronger than just being $\mathcal{B}[0,\infty) \otimes \mathcal{F} \to \mathcal{B}$ measurable: it also incorperates adaptedness.

Lemma: Let $\varphi : [0,\infty) \times \Omega \to S$; $\varphi(t,\omega) = X_t(\omega)$ (so $\varphi^T = \varphi|_{[0,T]\times\Omega}$). If $X.$ is progressively measurable, then it is adapted, and φ is $\mathcal{B}[0,\infty) \otimes \mathcal{F} \to \mathcal{B}$ measurable.

Pf. Set $\eta_T(\omega) = (T, \omega) \in [0,T] \times \Omega$

Exercise.

Check that η_T is $\mathcal{F}_T \to \mathcal{B}[0,T] \otimes \mathcal{F}_T$ measurable.

$\therefore$ $X_T = \varphi^T \circ \eta_T$ is $\mathcal{F}_T \to \mathcal{B}$ measurable

$\therefore$ $X.$ is adapted.

Now we must show that $\varphi : [0,\infty) \times \Omega \to S$ is $\mathcal{B}[0,\infty) \otimes \mathcal{F} \to \mathcal{B}$ measurable.

Take $V \in \mathcal{B}$. Then $\varphi^{-1}(V) \cap ([0,T] \times \Omega)$

$$= \varphi_T^{-1}(V) \in \mathcal{B}[0,T] \otimes \mathcal{F}_T \subseteq \mathcal{B}[0,\infty) \otimes \mathcal{F}$$

Now, $[0,\infty) = \bigcup_{T \in \mathbb{N}} [0,T]$,

$$\therefore \varphi^{-1}(V) = \bigcup_{T \in \mathbb{N}} \varphi^{-1}(V) \cap ([0,T] \times \Omega) \quad \therefore \in \mathcal{B}[0,\infty) \otimes \mathcal{F}.$$

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It is possible for a process to be adapted but not progressively measurable. Fortunately, this pathology doesn't occur in processes we care about.

Prop: Suppose S is a separable metric space, and

$$X_t : (\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathcal{F}, \mathbb{P}) \to (S, \mathcal{B}(S))$$

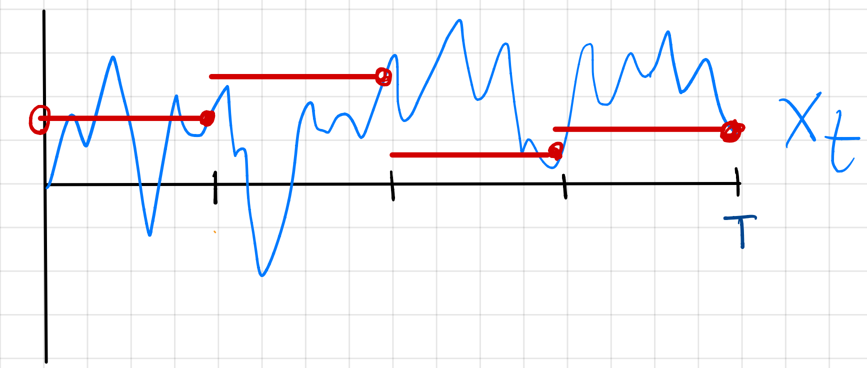
is adapted and right-continuous.

Then $X_\cdot$ is progressively measurable.

The idea is to approximate $X_\cdot$ by piecewise constant processes that are easily seen to be progressively measurable and use robustness of measurability under limits.

Pf. Approximate the function $\varphi^T(t,\omega) = X_t(\omega)$, $(t,\omega) \in [0,T] \times \Omega$ by

$$\varphi_n(t,\omega) = \sum_{k=1}^{2^n} X_{\frac{kT}{2^n}}(\omega)\, \mathbb{1}_{\left(\frac{(k-1)T}{2^n}, \frac{kT}{2^n}\right]}(t).$$



Then $\lim_{n\to\infty} \varphi_n(t,\omega) = \varphi^T(t,\omega)$ for $(t,\omega) \in [0,T) \times \Omega$

For any $V \in \mathcal{B}$,

$$\varphi_n^{-1}(V) = \underbrace{\left(\{0\} \times X_0^{-1}(V)\right) \cup \bigcup_{k=1}^{2^n} \left(\left(\frac{(k-1)T}{2^n}, \frac{kT}{2^n}\right] \times \underbrace{X_{\frac{kT}{2^n}}^{-1}(V)}_{\in\, \mathcal{F}_{\frac{kT}{2^n}}}\right)}_{\in\, \mathcal{B}[0,\frac{kT}{2^n}] \otimes \mathcal{F}_{\frac{kT}{2^n}} \,\subseteq\, \mathcal{B}[0,T] \otimes \mathcal{F}_T}.$$

$\therefore$ φ_n is $\mathcal{B}[0,T] \otimes \mathcal{F}_T \to \mathcal{B}$ measurable $\forall n \in \mathbb{N}$,

and $\therefore$ so too is $\varphi^T = \lim_{n\to\infty} \varphi_n$. ///