The characterization of Brownian motion as the unique centered continuous Gaussian process with covariance $\mathbb{E}[B_s B_t] = s \wedge t$ is very useful for recognizing Brownian motion in sometimes unexpected places.

Theorem: Let $B = (B_t)_{t \geq 0}$ be a Brownian motion. The following processes are also Brownian motions

(a) $R_t = -B_t$. (reflection)

(b) $D_t = c^{-1/2} B_{ct}$ for any $c > 0$. (diffusion scaling)

(c) $C_t = t B_{1/t}$ for $t > 0$, $C_0 = 0$. (time inversion)

(d) $S_t = B_{t+T} - B_T$ (time shifted)

(e) $M_t = B_{T-t} - B_T$ (time reversed)

for any $T > 0$. $t \in [0, T]$

Pf. In all cases, it is easy to check that the process is Gaussian, and centered.
In (a), (b), (d), (e) continuity is also clear, and so the result follows by calculating the covariance, which is routine.

We'll focus on time inversion (c).

$$C_t = \begin{cases} tB_{1/t}, & t>0 \\ 0, & t=0 \end{cases}$$

- Gaussian:

- Mean and covariance:

- Continuity: (only $t=0$ is unclear)

Cor: (BM LLN) Let $(B_t)_{t\geq 0}$ be a Brownian motion. Let $\beta > 0$. Then:

$$\limsup_{t\to\infty} \frac{|B_t|}{t^\beta} = \begin{cases} 0 & \text{if } \beta > \frac{1}{2} \\ \infty & \text{if } 0 < \beta < \frac{1}{2} \end{cases} \quad \text{a.s.}$$

Pf. Let $C_t = tB_{1/t}$. Since $C.$ is a Brownian motion, $(C_t)_{t\in[0,1]}$ is a.s. C^α for $\alpha < \frac{1}{2}$.

OTOH, per [Lec 54.2], $\limsup_{s\to 0} \frac{|C_s|}{s^\alpha} = \infty$ a.s. if $\alpha > \frac{1}{2}$.

The exact rate of divergence of Brownian motion as $t \to \infty$ is known.

Theorem: (Khinchin, Kolmogorov; Hartman-Wintner)

The Law of the Iterated Logarithm

$$\limsup_{t \to \infty} \frac{\pm B_t}{\sqrt{2t \log(\log t)}} = 1 \quad \text{a.s.}$$

Using Donsker's CLT, this can be used to show: if $(X_n)_{n \geq 1}$ are iid L^2 standardized,

$$\limsup_{n \to \infty} \frac{\pm S_n}{\sqrt{2n \log(\log n)}} = 1 \quad \text{a.s.}$$

Note: by the classic CLT,