The characterization of Brownian motion as the unique centered continuous Gaussian process with covariance $\mathbb{E}[B_s B_t] = s \wedge t$ is very useful for recognizing Brownian motion in sometimes unexpected places.

Theorem: Let $B = (B_t)_{t \geq 0}$ be a Brownian motion. The following processes are also Brownian motions

(a) $R_t = -B_t$. (reflection)

(b) $D_t = c^{-1/2} B_{ct}$ for any $c > 0$. (diffusion scaling)

(c) $C_t = t B_{1/t}$ for $t > 0$, $C_0 = 0$. (time inversion)

(d) $S_t = B_{t+T} - B_T$ ← indep. from $\mathcal{F}_T^B$. (time shifted)

for any $T > 0$.

(e) $M_t = B_{T-t} - B_T$ ← $t \in [0, T]$ (time reversed)

Pf. In all cases, it is easy to check that the process is Gaussian, and centered.
In (a), (b), (d), (e) continuity is also clear, and so the result follows by calculating the covariance, which is routine.

We'll focus on time inversion (c).

$$C_t = \begin{cases} tB_{1/t}, & t>0 \\ 0, & t=0 \end{cases}$$

$s_j := \begin{cases} 1/t_j & t_j>0 \\ 0 & t_j=0 \end{cases}$

- Gaussian: $\begin{bmatrix} C_{t_1} \\ C_{t_2} \\ \vdots \\ C_{t_n} \end{bmatrix} = \begin{bmatrix} t_1 B_{1/t_1} \\ t_2 B_{1/t_2} \\ \vdots \\ t_n B_{1/t_n} \end{bmatrix} = \begin{bmatrix} t_1 & & 0 \\ & t_2 & \\ 0 & & \ddots \; t_n \end{bmatrix} \begin{bmatrix} B_{s_1} \\ B_{s_2} \\ \vdots \\ B_{s_n} \end{bmatrix}$ ✓

- Mean and covariance: $\mathbb{E}[C_t] = t\,\mathbb{E}[B_{1/t}] = 0.$

$\mathbb{E}[C_s C_t] = \mathbb{E}[sB_{1/s} \cdot tB_{1/t}] = st\,\mathbb{E}[B_{1/s} B_{1/t}]$

$s \le t \therefore \frac{1}{s} \ge \frac{1}{t}$ $\longrightarrow$ $= st \left(\frac{1}{s}\right) \wedge \left(\frac{1}{t}\right)$

$= st \cdot \frac{1}{t} = s = s \wedge t.$ ✓

- Continuity: (only $t=0$ is unclear)

$(C_t)_{t\ge 0}$ is a pre-BM $\therefore$ By Kolmogorov,
$\exists\, (\tilde{C}_t)_{t\ge 0}$ cont. version,
$\therefore$ $C, \tilde{C}$ indistinguishable. ✓ ///

Cor: (BM LLN) Let $(B_t)_{t\geq 0}$ be a Brownian motion. Let $\beta > 0$. Then:

$$\limsup_{t\to\infty} \frac{|B_t|}{t^\beta} = \begin{cases} 0 & \text{if } \beta > \frac{1}{2} \\ \infty & \text{if } 0 < \beta < \frac{1}{2} \end{cases} \quad \text{a.s.}$$

Pf. Let $C_t = tB_{1/t}$. Since $C.$ is a Brownian motion, $(C_t)_{t\in[0,1]}$ is a.s. C^α for $\alpha < \frac{1}{2}$.

$$\exists K_\alpha < \infty \text{ a.s. s.t. } |C_s - \cancel{C_0}| \leq K_\alpha |s-0|^\alpha \ \forall s\in[0,1].$$

$$\therefore \frac{1}{t}|B_t| \leq K_\alpha t^{-\alpha} \xleftarrow[t=1/s]{} |sB_{1/s}| \leq K_\alpha s^\alpha \quad s\in[0,1]$$

$$\forall t \geq 1$$

$$|B_t| \leq K_\alpha t^{1-\alpha}$$

$$\therefore |B_t| \leq K_\alpha t^{\beta-\varepsilon}$$

$$\frac{|B_t|}{t^\beta} \leq K_\alpha t^{-\varepsilon} \quad \forall t \geq 1$$

If $\beta > \frac{1}{2}$, $\exists \varepsilon > 0$ s.t. $\beta - \varepsilon > \frac{1}{2}$. Define $\alpha := 1-(\beta-\varepsilon) < \frac{1}{2}$

OTOH, per [Lec 54.2], $\limsup_{s\to 0} \frac{|C_s|}{s^\alpha} = \infty$ a.s. if $\alpha > \frac{1}{2}$. chose $\alpha = 1-\beta$

$t = \frac{1}{s}$

$$\infty = \limsup_{t\to\infty} \frac{|C_{1/t}|}{t^{-\alpha}}$$

$$= \limsup_{t\to\infty} t^\alpha \left|\frac{1}{t}B_t\right| = \limsup_{t\to\infty} \frac{|B_t|}{t^{1-\alpha}}$$

$$= \limsup_{t\to\infty} \frac{|B_t|}{t^\beta} \quad ///$$

The exact rate of divergence of Brownian motion as $t \to \infty$ is known.

Theorem: (Khinchin, Kolmogorov; Hartman-Wintner)

The Law of the Iterated Logarithm

$$\limsup_{t\to\infty} \frac{\pm B_t}{\sqrt{2t\log(\log t)}} = 1 \quad \text{a.s.}$$

Using Donsker's CLT, this can be used to show: if $(X_n)_{n\geq 1}$ are iid L^2 standardized,

$$\limsup_{n\to\infty} \frac{\pm S_n}{\sqrt{2n\log(\log n)}} = 1 \quad \text{a.s.}$$

Note: by the classic CLT, $\quad \frac{S_n}{\sqrt{n}} \to_w \mathcal{N}(0,1)$

$$\Rightarrow \frac{S_n}{\sqrt{n}\cdot\sqrt{2\log\log n}} \to_{\mathbb{P}} 0.$$

$$\left|\frac{S_n}{\sqrt{2n\log\log n}}\right| < \varepsilon \quad \underline{\text{w}} \quad \mathbb{P} \to 1 \text{ as } n\to\infty, \ \forall \varepsilon>0.$$

$> \varepsilon$ i.o. for any $\varepsilon < 1$, a.s.