We've constructed Brownian motion $(B_t)_{t\in[0,T]}$ for any $T>0$; in other words, we have the Wiener measures:

$$W_T^x \in \mathrm{Prob}(C([0,T],\mathbb{R}^d))$$

It is easy to put these together and let $T\to\infty$.

↳ Start with pre-BM, Markov process $(X_t)_{t\geq 0}$ ($X_t - X_s \overset{d}{=} N(0,t-s)$, indep. from $\mathcal{F}_s^X$)

↳ Using Kolmogorov's Continuity Criterion, find a version $(\tilde{B}^T_t)_{0\leq t\leq T}$ that is continuous, for each $T\in\mathbb{N}$.

↳ $\tilde{B}^{T+1}|_{[0,T]}$, $\tilde{B}^T$ are both versions of $X|_{[0,T]}$, so are versions of each other. Both continuous, ∴ indistinguishable [HW].

$$\mathbb{P}(\exists t\in[0,T] : \tilde{B}^T_t \neq \tilde{B}^{T+1}_t) = 0$$

↳ ∴ For $t\in[0,\infty)$, define $B_t :=$

Let $W^x = \mathrm{Law}(B_t) \in \mathrm{Prob}(C[0,\infty),\mathbb{R}^d))$

→ Paths are $C^\alpha[0,T]$ $\forall T<\infty$, $\alpha<\frac{1}{2}$

→ Paths are not locally C^α at any point if $\alpha\geq\frac{1}{2}$

$\Big\}$ w/ $\mathbb{P}=1$.

Covariance

Let $s<t$. Then

$\mathbb{E}[B_s B_t]$

So, in general, $\mathbb{E}[B_s B_t]$

Note: if $\{X_t\}_{t\in T}$ is <u>any</u> collection of random variables on a given probability space, the function $\chi: T\times T \to \mathbb{R}$, $\chi(s,t) = \mathrm{Cov}[X_s, X_t]$ has a positivity property.

<u>Def</u>: A function $\chi: T\times T \to \mathbb{R}$ is positive definite iff for any finite subset $\Lambda = \{t_1, \dots, t_n\} \subseteq T$, the matrix $M_{ij} = \chi(t_i, t_j)$ is positive semidefinite.

Lemma: If $(X_t)_{t\in T} : (\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ are random variables, then

$\chi(t,s) = \chi(s,t) = \mathrm{Cov}[X_s, X_t]$ is positive-definite.

Pf. Fix $\Lambda \subseteq T$ finite, and note that for any $\xi : \Lambda \to \mathbb{R}$,

$$\sum_{s,t\in T} \chi(s,t)\,\xi(s)\,\xi(t)$$

So, we now know that the function

$\chi(s,t) = s \wedge t$ is positive definite.

Eg. $\chi(s,t) = s\wedge t - st, \quad 0 \leq s,t \leq 1.$

Gaussian Processes

Recall [Lec 26.1] a random vector $\mathbf{X} \in \mathbb{R}^d$ is called (jointly) Gaussian if the characteristic function has the form

$$\varphi_{\mathbf{X}}(\xi) = \mathbb{E}[e^{i\xi\cdot\mathbf{X}}] = e^{-\frac{1}{2}\|A^T\xi\|^2}$$

for some $A \in M_{d\times d}$.

Equivalently, by the Cramér-Wold device,

$\mathbf{X}$ is Gaussian iff $\xi\cdot\mathbf{X}$ is a normal random variable $\forall \xi \in \mathbb{R}^d$.

It is not sufficient just to check that the components of $\mathbf{X}$ are normally distributed.

Eg. $X \stackrel{d}{=} N(0,1)$, $R \stackrel{d}{=} \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}$, X, R independent.

$Y = RX$

But (X,Y) is not jointly Gaussian. [HW]

Note: if $T:\mathbb{R}^d \to \mathbb{R}^d$ is an invertible linear transformation, and if $\underline{Y} = T(\underline{X})$ is a Gaussian random vector, then so is $\underline{X}$.

In particular: permuting the entries preserves joint Gaussianness.

Def: A stochastic process $(X_t)_{t\in T}: (\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ is called a Gaussian Process if, for any finite collection of times $t_1, \dots, t_n \in T$, $(X_{t_1}, \dots, X_{t_n})$ is a (jointly) Gaussian random vector.

Prop: Brownian motion is a Gaussian process.

Pf. Let $0 \leq t_1 < t_2 < \cdots < t_n$.

Let $T(x_1, x_2, \dots, x_n) = (x_1, x_2 - x_1, x_3 - x_2, \dots, x_n - x_{n-1})$

$$T(B_{t_1}, \dots, B_{t_n}) = (B_{t_1} \quad , B_{t_2} - B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}})$$

Theorem: Let $c: T \to \mathbb{R}$ be any function, and let $\chi: T \times T \to \mathbb{R}$ be pos. definite. Then there exists a Gaussian process $(X_t)_{t \in T}: (\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ with

$$\mathbb{E}[X_t] = c(t) \quad \text{and} \quad \mathrm{Cov}(X_s, X_t) = \chi(s,t) \quad \forall s,t \in T.$$

Moreover, any two Gaussian processes with mean c and covariance φ have the same finite-dimensional distributions.

Pf. Existence is an exercise in Kolmogorov's Extension theorem; [Driver, Prop 31.6].

For f.d. uniqueness: if $\mathbf{X}$ is a Gaussian vector,

$$\varphi_{\mathbf{X}}(\xi) = e^{-\frac{1}{2}\xi \cdot C \xi} \quad \text{for a positive semi-definite matrix } C.$$

$$e^{-\frac{1}{2}\xi \cdot C \xi} = e^{-\frac{1}{2}\sum_{a,b} C_{ab}\xi_a \xi_b} \quad \therefore \varphi_{\mathbf{X}} \text{ is determined by}$$

$$\mathbb{E}[\mathbf{X}] \text{ and } \mathrm{Cov}\,\mathbf{X}.$$

Cor: If $(X_t)_{t\in[0,\infty)}$ is a continuous Gaussian process with

$$\mathbb{E}[X_t]=0, \quad \mathbb{E}[X_sX_t]=s\wedge t \qquad \forall s,t\geq 0$$

then $X_\cdot$ is a Brownian motion.

Pf. By the uniqueness result of the last theorem, $X_\cdot$ and $B_\cdot$ have the same finite-dimensional distributions.