We've constructed Brownian motion $(B_t)_{t\in[0,T]}$ for any $T>0$; in other words, we have the Wiener measures:

$$W^x_T \in \mathrm{Prob}(C([0,T],\mathbb{R}^d))$$

It is easy to put these together and let $T\to\infty$.

↳ Start with pre-BM, Markov process $(X_t)_{t\geq 0}$ ($X_t - X_s \overset{d}{=} N(0,t-s)$, indep. from $\mathcal{F}^X_s$)

↳ Using Kolmogrov's Continuity Criterion, find a version $(\tilde{B}^T_t)_{0\leq t\leq T}$ that is continuous, for each $T\in\mathbb{N}$.

↳ $\tilde{B}^{T+1}|_{[0,T]}$, $\tilde{B}^T$ are both versions of $X|_{[0,T]}$, so are versions of each other. Both continuous, $\therefore$ indistinguishable [HW].

$$\mathbb{P}(\exists t\in[0,T] : \tilde{B}^T_t \neq \tilde{B}^{T+1}_t) = 0$$

↳ $\therefore$ For $t\in[0,\infty)$, define $B_t := \tilde{B}^{\lceil t\rceil}_t$. well-defined continuous.

Let $W^x = \mathrm{Law}(B_t) \in \mathrm{Prob}(C[0,\infty),\mathbb{R}^d))$

→ Paths are $C^\alpha[0,T]$ $\forall T<\infty$, $\alpha<\frac{1}{2}$

→ Paths are not locally C^α at any point if $\alpha\geq\frac{1}{2}$

w/ $\mathbb{P}=1$.

Covariance

Let $s<t$. Then

$$\mathbb{E}[B_sB_t] = \mathbb{E}[\underbrace{B_s(B_s}+B_t-B_s)] = \underbrace{\mathbb{E}[B_s^2]}_{=\;s} + \mathbb{E}[\overset{B_s-B_0}{B_s}(B_t-B_s)]$$

$$\cancel{\mathbb{E}[B_s]\,\mathbb{E}[B_t-B_s]}$$

So, in general, $\mathbb{E}[B_sB_t] = s\wedge t$

$$\overset{\shortparallel}{\mathrm{Cov}}(B_s,B_t) \quad b/c \quad \mathbb{E}[B_s]=0 \ \forall s\geq 0.$$

Note: if $\{X_t\}_{t\in T}$ is <u>any</u> collection of random variables on a given probability space, the function $\chi: T\times T\to\mathbb{R}$, $\chi(s,t) = \mathrm{Cov}[X_s,X_t]$ has a positivity property.

<u>Def:</u> A function $\chi: T\times T\to\mathbb{R}$ is positive ~~definite~~ semi-definite iff for any finite subset $\Lambda=\{t_1,\dots,t_n\}\subseteq T$, the matrix $M_{ij}=\chi(t_i,t_j)$ is positive semidefinite.

Ex. $M=M^T$ & $\xi\cdot M\xi\geq 0 \quad \forall \xi\in\mathbb{R}^n$ $\quad \forall \xi:\Lambda\to\mathbb{R}$.

$$\chi(s,t)=\chi(t,s) \quad \& \quad \sum_{s,t\in\Lambda}\chi(s,t)\xi(s)\xi(t)\geq 0$$

Lemma: If $(X_t)_{t\in T} : (\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ are random variables, then

$\chi(t,s) = \chi(s,t) = \mathrm{Cov}[X_s, X_t]$ is positive-definite.

Pf. Fix $\Lambda \subseteq T$ finite, and note that for any $\xi : \Lambda \to \mathbb{R}$,

$$\sum_{s,t\in T} \chi(s,t)\,\xi(s)\,\xi(t) = \sum_{s\in\Lambda}\sum_{t\in\Lambda} \xi(s)\,\xi(t)\,\mathrm{Cov}(X_s, X_t)$$

$$= \mathrm{Cov}\Big(\sum_{s\in\Lambda} \xi(s) X_s, \sum_{t\in\Lambda} \xi(t) X_t\Big)$$

Cov is bilinear

$$= \mathrm{Var}\Big[\sum_{t\in T} \xi(t) X_t\Big] \geq 0. \qquad ///$$

So, we now know that the function

$\chi(s,t) = s \wedge t$ is positive definite.

Eg. $\chi(s,t) = s\wedge t - st, \quad 0 \leq s,t \leq 1.$

Let $(B_t)_{t\in[0,1]}$ be a B.M. $B_0 = 0$

Define $X_t = B_t - tB_1.$ ← Brownian bridge.

$X_0 = 0, \; X_1 = B_1 - B_1 = 0.$

$$\mathrm{Cov}(X_s, X_t) = \mathbb{E}[(B_s - sB_1)(B_t - tB_1)]$$

$$= \mathbb{E}[B_s B_t] - t\,\mathbb{E}[B_s B_1] - s\,\mathbb{E}[B_t B_1] + st\,\mathbb{E}[B_1^2]$$

$$= s\wedge t - t(s\wedge 1) - s(t\wedge 1) + st\cdot 1$$

Gaussian Processes

Recall [Lec 26.1] a random vector $\mathbf{X} \in \mathbb{R}^d$ is called (jointly) Gaussian if the characteristic function has the form

$$\varphi_{\mathbf{X}}(\xi) = \mathbb{E}[e^{i\xi\cdot\mathbf{X}}] = e^{-\frac{1}{2}\|A^T\xi\|^2} = e^{-\frac{1}{2}\xi\cdot C\xi} \qquad C = AA^T \text{ pos. semidef}$$

$$\varphi_{\mathbf{X}}(\xi) = e^{-\frac{1}{2}\xi\cdot C\xi + i\mu\cdot\xi}$$

$$\mu = \mathbb{E}[\mathbf{X}].$$

for some $A \in M_{d\times d}$.

Equivalently, by the Cramér-Wold device,

$\mathbf{X}$ is Gaussian iff $\xi\cdot\mathbf{X}$ is a normal random variable $\forall \xi \in \mathbb{R}^d$.

Eg. $\xi = \vec{e}_k \Rightarrow X_k$ are normally distributed, $k \in \{1, \dots, d\}$.

It is not sufficient just to check that the components of $\mathbf{X}$ are normally distributed.

Eg. $X \stackrel{d}{=} N(0,1)$, $R \stackrel{d}{=} \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}$, X, R independent.

$Y = RX \stackrel{d}{=} N(0,1)$. $\qquad X \stackrel{d}{=} -X$

$$\mathbb{E}[f(Y)] = \mathbb{E}[f(RX)] = \mathbb{E}[f(X)\,|\,R=1]\underbrace{\mathbb{P}(R=1)}_{\frac{1}{2}} + \underbrace{\mathbb{E}[f(-X)\,|\,R=-1]}_{\mathbb{E}[f(X)]}\underbrace{\mathbb{P}(R=-1)}_{\frac{1}{2}}$$

$$= \mathbb{E}[f(X)] \quad \therefore X \stackrel{d}{=} Y.$$

But (X, Y) is not jointly Gaussian. [HW]

Note: if $T:\mathbb{R}^d \to \mathbb{R}^d$ is an invertible linear transformation, and if $\underline{Y} = T(\underline{X})$ is a Gaussian random vector, then so is $\underline{X}$.

$$\xi \cdot \underline{X} = \xi \cdot T^{-1}(\underline{Y}) = (T^{-1})^T \xi \cdot \underline{Y} \text{ is normally distributed.}$$

In particular: permuting the entries preserves joint Gaussianness.

Def: A stochastic process $(X_t)_{t\in T} : (\Omega, \mathcal{F}, P) \to \mathbb{R}$ is called a Gaussian Process if, for any finite collection of times $t_1, \dots, t_n \in T$, $(X_{t_1}, \dots, X_{t_n})$ is a (jointly) Gaussian random vector.

Prop: Brownian motion is a Gaussian process.

Pf. Let $0 \le t_1 < t_2 < \cdots < t_n$. (suffic. ↑)

Let $T(x_1, x_2, \dots, x_n) = (x_1, x_2 - x_1, x_3 - x_2, \dots, x_n - x_{n-1})$

inverse: $T^{-1}(y_1, \dots, y_n) = (y_1, y_1+y_2, y_1+y_2+y_3, \dots, y_1+\cdots+y_n)$

$T(B_{t_1}, \dots, B_{t_n}) = (B_{t_1} - B_0, B_{t_2} - B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}})$

$$\stackrel{d}{=} \mathcal{N}\left(\underline{0}, \begin{bmatrix} t_1 & & & 0 \\ & t_2 - t_2 & & \\ & & \ddots & \\ 0 & & & t_n - t_{n-1} \end{bmatrix}\right) \quad ///$$

Theorem: Let $c: T \to \mathbb{R}$ be any function, and let $\varphi : T\times T \to \mathbb{R}$ be pos. definite. Then there exists a Gaussian process $(X_t)_{t\in T} : (\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ with

$$\mathbb{E}[X_t] = c(t) \quad \text{and} \quad \mathrm{Cov}(X_s, X_t) = \varphi(s,t) \quad \forall s,t\in T.$$

Moreover, any two Gaussian processes with mean c and covariance φ have the same finite-dimensional distributions.

Pf. Existence is an exercise in Kolmogorov's Extension theorem; [Driver, Prop 31.6].

For f.d. uniqueness: if $\mathbf{X}$ is a Gaussian vector,

$$\varphi_{\mathring{\mathbf{X}}}(\xi) = e^{-\frac{1}{2}\xi\cdot C\xi} \quad \text{for a positive semi-definite matrix } C.$$

$$\frac{\partial^2}{\partial\xi_j\partial\xi_k}\mathbb{E}\left[e^{i\xi\cdot\mathring{\mathbf{X}}}\right]\Big|_{\xi=0} = \mathbb{E}\left[(i\mathring{X}_j)(i\mathring{X}_k)e^{i\xi\cdot\mathring{\mathbf{X}}}\right]\Big|_{\xi=0}$$
$$= -\mathbb{E}[\mathring{X}_j\mathring{X}_k] = -\mathrm{Cov}(X_j, X_k)$$

$$e^{-\frac{1}{2}\xi\cdot C\xi} = e^{-\frac{1}{2}\sum_{a,b}C_{ab}\xi_a\xi_b} \qquad \therefore\ \varphi_{\mathbf{X}} \text{ is determined by}$$

$$\frac{\partial^2}{\partial\xi_j\partial\xi_k}\Big(\qquad\Big)\Big|_{\xi=0} = -C_{jk}. \qquad \mathbb{E}[\mathbf{X}] \text{ and } \mathrm{Cov}\,\mathbf{X}.$$

///

Cor: If $(X_t)_{t\in[0,\infty)}$ is a continuous Gaussian process with

$$\mathbb{E}[X_t]=0, \quad \mathbb{E}[X_s X_t] = s\wedge t \quad \forall s,t\geq 0$$

then $X_\cdot$ is a Brownian motion.

Pf. By the uniqueness result of the last theorem, $X_\cdot$ and $B_\cdot$ have the same finite-dimensional distributions. ///