

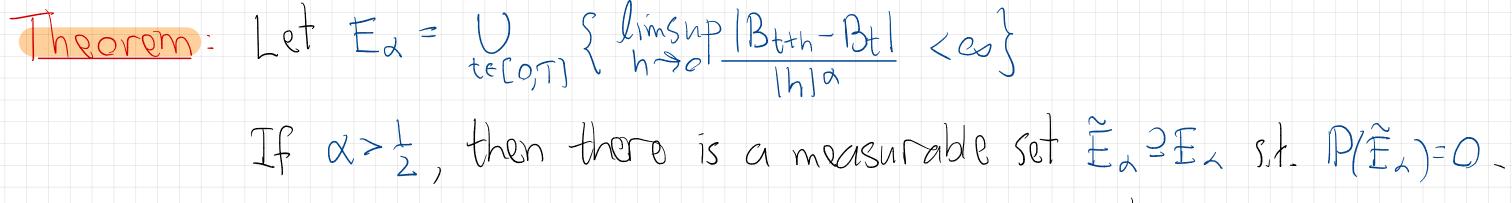
Note: the ratio is finite for any sit; thus, an equivalent formulation co > limsup ||w(s)-w(t)|| $s > t ||s-t|^{\alpha}$

If a path is Ca on [gT], then it is locally Ca at every trigT; the Converse is not true. In fact WE Carojt iff

In the last lecture, we slowed that, with $\alpha > \frac{1}{2}$,

Brawnian motion is a.s. not C^2 . I.S. $P(Sup limsup Bth-Bt < \infty) = 0$ P(tergT) has P(tergT)

This desn't preclude the possibility that BM is locally C2, perhaps even at every point. But that's not true.



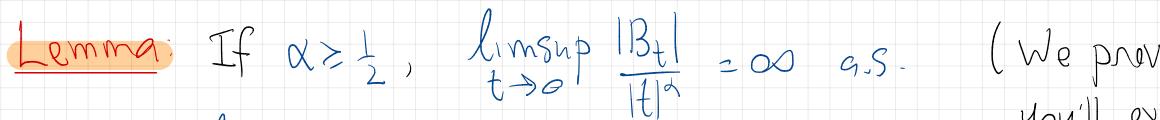
I.e., P*(Ea)=0, "Brownian motion is nowhere locally Cd, y P=1,"

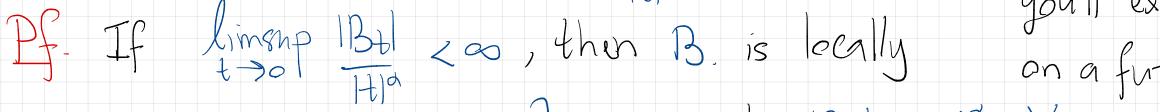
Cor Brownian motion is nowhere differentiable, a.s.

Pf. If tr> Bytw) is diffible at some point t, then for any a ((21),

linsup IBE+h(w)-BE(w) h=>0 1h1x

As a first step to the proof, we show Brownian motion is not locally Ca at t=0 for d>1.



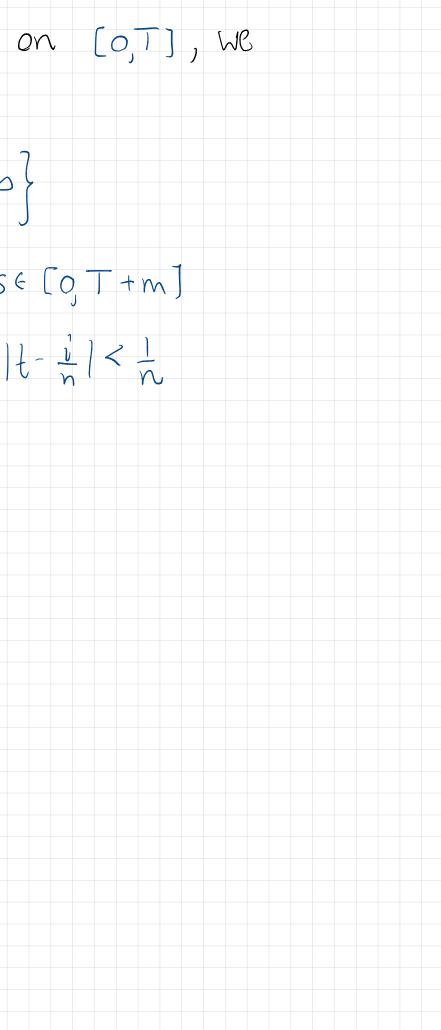


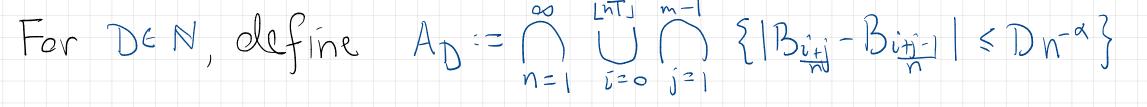
Ca @ t=0, and so I C<00 s.t. IBy I < Cta Vbe Co, TI,

(We prove this for $d > \frac{1}{2}$; you'll explore the d= 1 GSE on a future (HW].)

Prof of Theorem: To prove B. is nowhere locally Ca (x>2) on [0,T], we will work with B. defined on a larger interval [0, T+m] Ex = {w: Itero, TJ, linsup Bt+h(w)-Btw) < co} So, for wEEx, JC < 00 s.t. |By(w) - Bs(w) | S CII-SIX VSE[0, T+m] · Approximate t by nationals: for any n, ZI iEN s.t. It-il< in Look@ Bs for s≈t+ tijil s= itt Solong as j < m-1, inth $|B_{i+j}(w) - B_{i+j-1}(w)|$ That is, 3D s,t. on Ez,

 $|B_{i+j} - B_{i+j-1}| \leq Dn^{-\alpha}$, $\forall n \in \mathbb{N}$, $|\leq j < m$





We just showed that Ex S U AD

Claim: YDEN, P(AD)=0.

To prove this, we make the same observation as in the lemma: $P(A_D) \leq \liminf_{n \to \infty} P(\bigcup_{i=0}^{n} \sum_{j=1}^{n-1} B_{ij} - B_{ij} \leq D_n - x_j)$ Note: $n \to \infty P(A_D) \leq \lim_{n \to \infty} P(\bigcup_{i=0}^{n} \sum_{j=1}^{n-1} B_{ij} - B_{ij} \leq D_n - x_j)$

 $\leq \liminf_{n \to \infty} \sum_{i=0}^{LnT_j} \frac{m-i}{j-i} P(|B_{i+j} - B_{i+j-1}| \leq Dn^{-\alpha})$

Note: the events $\left\{ \left| B_{\hat{v}+\hat{j}} - B_{\hat{v}+\hat{j}-1} \right| \leq D_{N} - \alpha \right\}_{\hat{j}=1}^{m-1}$ are independent

We've shown that $P(A_D) \leq (T+I) \liminf_{n \to \infty} n \cdot P(|z| \leq Dn^{\frac{1}{2}-a})^{m-1}$ where $Z \stackrel{q}{=} N(q_i)$

Note: this shows that $(B_t)_{t \in (0, T+m]}$ is a.s. not locally $C^{\alpha}[0,T]$ for any $\alpha > \frac{1}{2}$ Since T was arbitrary, this preves the paths are

rough everywhere.

