Def: A path $\omega \in C([0,T], S)$ is (locally) C^α at $t \in [0,T]$ if

$$\sup_{s \in [0,T]} \frac{\|\omega(s) - \omega(t)\|}{|s-t|^\alpha} < \infty$$

Note: the ratio is finite for any $s \neq t$; thus, an equivalent formulation is

$$\infty > \limsup_{s \to t} \frac{\|\omega(s) - \omega(t)\|}{|s-t|^\alpha}$$

If a path is C^α on $[0,T]$, then it is locally C^α at every $t \in [0,T]$; the converse is not true. In fact

$\omega \in C^\alpha[0,T]$ iff

In the last lecture, we showed that, with $\alpha > \frac{1}{2}$, Brownian motion is a.s. not C^α. I.e.

$$\mathbb{P}\left(\sup_{t \in [0,T]} \limsup_{h \to 0} \frac{B_{t+h} - B_t}{|h|^\alpha} < \infty \right) = 0$$

This doesn't preclude the possibility that BM is locally C^α, perhaps even at every point. But that's not true.

Theorem: Let $E_\alpha = \bigcup_{t\in[0,T]} \left\{ \limsup_{h\to 0} \frac{|B_{t+h}-B_t|}{|h|^\alpha} < \infty \right\}$

If $\alpha > \frac{1}{2}$, then there is a measurable set $\tilde{E}_\alpha \supseteq E_\alpha$ s.t. $\mathbb{P}(\tilde{E}_\alpha) = 0$.

I.e. $\mathbb{P}^*(E_\alpha) = 0$, "Brownian motion is nowhere locally C^α, w $\mathbb{P}=1$."

Cor: Brownian motion is nowhere differentiable, a.s.

Pf. If $t \mapsto B_t(\omega)$ is diff'ble at some point t, then for any $\alpha \in (\frac{1}{2}, 1)$,

$$\limsup_{h\to 0} \frac{|B_{t+h}(\omega) - B_t(\omega)|}{|h|^\alpha}$$

As a first step to the proof, we show Brownian motion is not locally C^α at $t=0$ for $\alpha > \frac{1}{2}$.

Lemma: If $\alpha \geq \frac{1}{2}$, $\limsup_{t\to 0} \frac{|B_t|}{|t|^\alpha} = \infty$ a.s. (We prove this for $\alpha > \frac{1}{2}$; you'll explore the $\alpha = \frac{1}{2}$ case on a future [HW].)

Pf. If $\limsup_{t\to 0} \frac{|B_t|}{|t|^\alpha} < \infty$, then $B.$ is locally C^α @ $t=0$, and so $\exists\, C < \infty$ s.t. $|B_t| \leq C t^\alpha$ $\forall t \in [0,T]$.

Proof of Theorem: To prove $B_\cdot$ is nowhere locally C^α $(\alpha > \frac{1}{2})$ on $[0,T]$, we will work with $B_\cdot$ defined on a larger interval $[0, T+m]$

$$E_\alpha = \left\{ \omega : \exists\, t \in [0,T],\ \limsup_{h \to 0} \frac{B_{t+h}(\omega) - B_t(\omega)}{|h|^\alpha} < \infty \right\}$$

So, for $\omega \in E_\alpha$, $\exists\, C < \infty$ s.t. $|B_t(\omega) - B_s(\omega)| \leq C|t-s|^\alpha$ $\forall s \in [0, T+m]$

· Approximate t by rationals: for any n, $\exists!\, i \in \mathbb{N}$ s.t. $|t - \frac{i}{n}| < \frac{1}{n}$

Look @ B_s for $s \approx t + \frac{j}{n}$; i.e. $s = \frac{i}{n} + \frac{j}{n}$

So long as $j \leq m-1$, $\frac{i}{n} + \frac{j}{n}$

$|B_{\frac{i+j}{n}}(\omega) - B_{\frac{i+j-1}{n}}(\omega)|$

That is, $\exists\, D$ s.t. on E_α,

$$|B_{\frac{i+j}{n}} - B_{\frac{i+j-1}{n}}| \leq D n^{-\alpha}, \quad \forall n \in \mathbb{N},\ 1 \leq j < m.$$

For $D \in \mathbb{N}$, define $A_D := \bigcap_{n=1}^{\infty} \bigcup_{i=0}^{\lfloor nT \rfloor} \bigcap_{j=1}^{m-1} \{ |B_{\frac{i+j}{n}} - B_{\frac{i+j-1}{n}}| \leq D n^{-\alpha} \}$.

We just showed that $E_\alpha \subseteq \bigcup_{D \in \mathbb{N}} A_D$

Claim: $\forall D \in \mathbb{N}$, $\mathbb{P}(A_D) = 0$.

To prove this, we make the same observation as in the lemma:

$$\mathbb{P}(A_D) \leq \liminf_{n \to \infty} \mathbb{P}\left(\bigcup_{i=0}^{\lfloor nT \rfloor} \bigcap_{j=1}^{m-1} \{ |B_{\frac{i+j}{n}} - B_{\frac{i+j-1}{n}}| \leq D n^{-\alpha} \} \right)$$

Note: the events $\{ |B_{\frac{i+j}{n}} - B_{\frac{i+j-1}{n}}| \leq D n^{-\alpha} \}_{j=1}^{m-1}$ are independent

$$\leq \liminf_{n \to \infty} \sum_{i=0}^{\lfloor nT \rfloor} \prod_{j=1}^{m-1} \mathbb{P}\left(|B_{\frac{i+j}{n}} - B_{\frac{i+j-1}{n}}| \leq D n^{-\alpha} \right)$$

we've shown that $\mathbb{P}(A_D) \leq (T+1) \liminf_{n\to\infty} n \cdot \mathbb{P}(|Z| \leq D n^{\frac{1}{2}-\alpha})^{m-1}$ where $Z \overset{d}{=} \mathcal{N}(0,1)$

Note: this shows that $(B_t)_{t\in[0,T+m]}$ is a.s. not locally $C^\alpha[0,T]$ for any $\alpha > \frac{1}{2}$

Since T was arbitrary, this proves the paths are rough everywhere.