Def: A path $\omega \in C([0,T], S)$ is (locally) C^α at $t \in [0,T]$ if

$$\sup_{\substack{s\in[0,T]\\ s\neq t}} \frac{\|\omega(s)-\omega(t)\|}{|s-t|^\alpha} < \infty$$

I.e. $\exists\, C_t < \infty$ s.t. $\|\omega(s)-\omega(t)\| \leq C_t |t-s|^\alpha \quad \forall s \in [0,T]$.

Note: the ratio is finite for any $s \neq t$; thus, an equivalent formulation is

$$\infty > \limsup_{s\to t} \frac{\|\omega(s)-\omega(t)\|}{|s-t|^\alpha} = \limsup_{h\to 0} \frac{\|\omega(t+h)-\omega(t)\|}{|h|^\alpha}$$

If a path is C^α on $[0,T]$, then it is locally C^α at every $t\in[0,T]$; the converse is not true. In fact

$$\omega \in C^\alpha[0,T] \quad \text{iff} \quad \sup_{t\in[0,T]} \limsup_{h\to 0} \frac{\|\omega(t+h)-\omega(t)\|}{|h|^\alpha} < \infty.$$

In the last lecture, we showed that, with $\alpha > \frac{1}{2}$, Brownian motion is a.s. not C^α. I.e.

$$\mathbb{P}\left(\sup_{t\in[0,T]} \limsup_{h\to 0} \frac{B_{t+h}-B_t}{|h|^\alpha} < \infty \right) = 0$$

This doesn't preclude the possibility that BM is locally C^α, perhaps even at every point. But that's not true.

Theorem: Let $E_\alpha = \bigcup_{t\in[0,T]} \left\{ \limsup_{h\to 0} \frac{|B_{t+h} - B_t|}{|h|^\alpha} < \infty \right\} = \left\{ \inf_t \limsup_{h\to 0} \frac{|B_{t+h}-B_t|}{|h|^\alpha} < \infty \right\}$

If $\alpha > \frac{1}{2}$, then there is a measurable set $\tilde{E}_\alpha \supseteq E_\alpha$ s.t. $\mathbb{P}(\tilde{E}_\alpha) = 0$.

I.e. $\mathbb{P}^*(E_\alpha) = 0$. "Brownian motion is nowhere locally C^α, w $\mathbb{P}=1$."

Cor: Brownian motion is nowhere differentiable, a.s.

Pf. If $t \mapsto B_t(\omega)$ is diff'ble at some point t, then for any $\alpha \in (\frac{1}{2}, 1)$,

$$\limsup_{h\to 0} \frac{|B_{t+h}(\omega) - B_t(\omega)|}{|h|^{\alpha}\,|h|}\,|h|^{1-\alpha}$$

$$= \underbrace{\limsup_{h\to 0} \left| \frac{B_{t+h}(\omega) - B_t(\omega)}{h} \right|}\ \limsup_{h\to 0} |h|^{1-\alpha}$$

$$= |B_t'(\omega)| \cdot 0 = 0.$$

$\therefore\ \omega \in E_\alpha \subseteq \tilde{E}_\alpha$

///

As a first step to the proof, we show Brownian motion is not locally C^α at $t=0$ for $\alpha > \frac{1}{2}$.

Lemma: If $\alpha \geq \frac{1}{2}$, $\limsup_{t\to 0} \frac{|B_t|}{|t|^\alpha} = \infty$ a.s. (We prove this for $\alpha > \frac{1}{2}$; you'll explore the $\alpha = \frac{1}{2}$ case on a future [HW].)

Pf. If $\limsup_{t\to 0} \frac{|B_t|}{|t|^\alpha} < \infty$, then $B_\cdot$ is locally C^α @ $t=0$, and so $\exists\, C < \infty$ s.t. $|B_t| \leq C t^\alpha \;\; \forall t \in [0,T]$.

$\therefore |B_{1/n}| \leq C n^{-\alpha} \;\; \forall$ large n $(n > \frac{1}{T})$

$$\left\{ \limsup_{t\to 0} \frac{|B_t|}{|t|^\alpha} < \infty \right\} \subseteq \bigcup_{C \in \mathbb{N}} \bigcap_{n > \frac{1}{T}} \{ |B_{1/n}| \leq C n^{-\alpha} \}$$

$$\mathbb{P}\Big(\underbrace{\bigcap_{n>\frac{1}{T}} \{ |B_{1/n}| \leq C n^{-\alpha} \}}_{\subseteq \bigcap_{n>\frac{1}{T}} \bigcup_{k \geq n} \{ |B_{1/k}| \leq C k^{-\alpha} \}} \Big) \leq \liminf_{n\to\infty} \mathbb{P}(|B_{1/n}| \leq C n^{-\alpha})$$

$$= \liminf_{n\to\infty} \mathbb{P}\left(\tfrac{1}{\sqrt{n}} |Z| \leq C n^{-\alpha}\right)$$

$$= \liminf_{n\to\infty} \mathbb{P}(|Z| \leq C n^{\frac{1}{2}-\alpha})$$

$$= \mathbb{P}(|Z| = 0) = 0.$$

///

Proof of Theorem: To prove $B.$ is nowhere locally C^α $(\alpha > \frac{1}{2})$ on $[0,T]$, we will work with $B.$ defined on a larger interval $[0, T+m]$

$$E_\alpha = \left\{ \omega : \exists\, t \in [0,T],\ \limsup_{h \to 0} \frac{B_{t+h}(\omega) - B_t(\omega)}{|h|^\alpha} < \infty \right\}$$

So, for $\omega \in E_\alpha$, $\exists\, C < \infty$ s.t. $|B_t(\omega) - B_s(\omega)| \leq C|t-s|^\alpha \quad \forall s \in [0, T+m]$

· Approximate t by rationals: for any n, $\exists!\, i \in \mathbb{N}$ s.t. $|t - \frac{i}{n}| < \frac{1}{n}$

Look @ B_s for $s \approx t + \frac{j}{n}$; i.e. $s = \frac{i}{n} + \frac{j}{n}$ $\qquad [0,T] = \bigsqcup_{k=1}^{\lfloor nT \rfloor} [\frac{k-1}{n}, \frac{k}{n}) \sqcup [\frac{\lfloor nT \rfloor}{n}, T]$

So long as $j \leq m-1$, $\frac{i}{n} + \frac{j}{n} \leq t + \frac{1}{n} + \frac{j}{n} \leq T + 1 + j \leq T + m$

$$|B_{\frac{i+j}{n}}(\omega) - B_{\frac{i+j-1}{n}}(\omega)|$$

$$\leq |B_{\frac{i+j}{n}}(\omega) - B_t(\omega)| + |B_t(\omega) - B_{\frac{i+j-1}{n}}(\omega)|$$

$$\leq C\underbrace{\left|\frac{i+j}{n} - t\right|^\alpha}_{|\frac{i}{n} - t + \frac{j}{n}| \leq \frac{j+1}{n}} + C\underbrace{\left|t - \frac{i+j-1}{n}\right|^\alpha}_{\leq \frac{j}{n}} \leq C\left(\left(\frac{j+1}{n}\right)^\alpha + \left(\frac{j}{n}\right)^\alpha\right)$$

$$\leq C n^{-\alpha}(2m^\alpha)$$

That is, $\exists\, D = 2m^\alpha C$ s.t. on E_α,

$$|B_{\frac{i+j}{n}} - B_{\frac{i+j-1}{n}}| \leq D n^{-\alpha}, \quad \forall n \in \mathbb{N},\ 1 \leq j < m.$$

For $D \in \mathbb{N}$, define $A_D := \bigcap_{n=1}^{\infty} \bigcup_{i=0}^{\lfloor nT \rfloor} \bigcap_{j=1}^{m-1} \{ |B_{\frac{i+j}{n}} - B_{\frac{i+j-1}{n}}| \leq D n^{-\alpha} \}$.

We just showed that $E_\alpha \subseteq \bigcup_{D \in \mathbb{N}} A_D =: \tilde{E}_\alpha$

Claim: $\forall D \in \mathbb{N}$, $\mathbb{P}(A_D) = 0$. $\therefore$ $\mathbb{P}^*(E_\alpha) \leq \mathbb{P}(\tilde{E}_\alpha) = 0$.

To prove this, we make the same observation as in the lemma:

$$\mathbb{P}(A_D) \leq \liminf_{n \to \infty} \underbrace{\mathbb{P}\left(\bigcup_{i=0}^{\lfloor nT \rfloor} \bigcap_{j=1}^{m-1} \{ |B_{\frac{i+j}{n}} - B_{\frac{i+j-1}{n}}| \leq D n^{-\alpha} \} \right)}_{\leq \sum_{i=0}^{\lfloor nT \rfloor} \mathbb{P}\left(\bigcap_{j=1}^{m-1} \text{''} \quad \text{''} \right)}$$

Note: the events $\{ |B_{\frac{i+j}{n}} - B_{\frac{i+j-1}{n}}| \leq D n^{-\alpha} \}_{j=1}^{m-1}$ are independent

$$\leq \liminf_{n \to \infty} \sum_{i=0}^{\lfloor nT \rfloor} \prod_{j=1}^{m-1} \mathbb{P}\left(|\underbrace{B_{\frac{i+j}{n}} - B_{\frac{i+j-1}{n}}}_{\overset{d}{=} N(0, \frac{1}{n}) \overset{d}{=} \frac{1}{\sqrt{n}} Z}| \leq D n^{-\alpha} \right) \qquad Z \overset{d}{=} N(0,1)$$

$$= \liminf_{n \to \infty} \underbrace{(\lfloor nT \rfloor + 1)}_{n(T+1)} \Big(\underbrace{\mathbb{P}(|\tfrac{1}{\sqrt{n}} Z| \leq D n^{-\alpha})}_{\mathbb{P}(|Z| \leq D n^{\frac{1}{2} - \alpha})} \Big)^{m-1}.$$

We've shown that $\mathbb{P}(A_D) \leqslant (T+1) \liminf_{n\to\infty} n \cdot \mathbb{P}(|Z| \leqslant Dn^{\frac{1}{2}-\alpha})^{m-1}$ where $Z \stackrel{d}{=} \mathcal{N}(0,1)$

$$\mathbb{P}(|Z| \leq \delta) = \int_{-\delta}^{\delta} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx \leq \int_{-\delta}^{\delta} \frac{1}{2} dx = \delta.$$

$$\leq (T+1) \liminf_{n\to\infty} n \cdot (Dn^{\frac{1}{2}-\alpha})^{m-1}$$

$$= (T+1) D^{m-1} \liminf_{n\to\infty} n^{1+(m-1)(\frac{1}{2}-\alpha)} = 0.$$

$\therefore$ Need $1+(m-1)(\frac{1}{2}-\alpha) < 0$

i.e. $\alpha > \frac{1}{2} + \frac{1}{m-1}$.

///

Note: this shows that $(B_t)_{t\in[0,T+m]}$ is a.s. not locally $C^\alpha[0,T]$ for any $\alpha > \frac{1}{2}$ $\left(m > 1 + \frac{1}{\alpha - \frac{1}{2}}\right)$

Since T was arbitrary, this proves the paths are rough everywhere.