Regularity of Paths

Let $(S, \| \, \|)$ be a normed space, and take $\omega \in C([0,T], S)$.
In general, let Π denote an interval partition of $[0,T]$

$$\Pi = \{0 = t_0 < t_1 < \cdots < t_n = T\} \in \mathcal{P}([0,T])$$

Def: Given $1 \le p < \infty$ and $\Pi \in \mathcal{P}([0,T])$, define

$$\mathcal{V}_p(\Pi, \omega) := \left(\sum_{j=1}^{n} \| \omega(t_j) - \omega(t_{j-1}) \|^p \right)^{1/p}$$

The **p-variation** $\mathcal{V}_p(\omega)$ of the path is $\mathcal{V}_p(\omega) := \sup_{\Pi \in \mathcal{P}([0,T])} \mathcal{V}_p(\Pi, \omega)$.

Eg. Suppose $\omega \in Lip([0,T])$. Then

$$\mathcal{V}_1(\Pi, \omega) = \sum_{j=1}^{n} \| \omega(t_j) - \omega(t_{j-1}) \|$$

If $\mathcal{V}_1(\omega) < \infty$, we say ω has **bounded variation**.

It is possible for a path to have unbounded variation, but still $v_p(w) < \infty$ for some $p > 1$. But if $v_p(w) < \infty$, $v_q(w) < \infty$ for $q > p$.

Prop For $w \in C([0,T], S)$, $p \mapsto v_p(w)$ is a decreasing function.

Pf. Follows from $p \mapsto v_p(\pi, w)$ being decreasing for any π.
For that, just note that for any $(a_j)_{j=1}^n \geq 0$ and $1 \leq p < q < \infty$

$$\sum_{j=1}^n a_j^q$$

In the $p=1$ case, there is an alternative way to compute $v_1(\omega)$.

$$v_1(\omega) = \sup_{\Pi \in \mathcal{P}([0,T])} v_1(\Pi, \omega)$$

The reason is the following:

Lemma: If $\Pi \subseteq \Pi'$ then $v_1(\Pi,\omega) \leq v_1(\Pi',\omega)$.

Pf. $\Pi = \{0 = t_0 < t_1 < t_2 < \cdots < t_n\}$

The same is not true for $p>1$.

Indeed, it is possible for $\lim_{n\to\infty} v_p(\Pi_n, \omega) < \infty$ but $v_p(\omega) = \infty$.

BV ($v_1 < \infty$) paths are precisely the Riemann-Stieltjes integrators:

$$\int f \, d\omega = \lim_{\|\Pi\| \to 0} \sum_{t_j \in \Pi} f(t_j^*) \left(\omega(t_j) - \omega(t_{j-1})\right).$$

Quadratic Variation

For $w \in C([0,T], S)$, $\Pi \in \mathcal{P}([0,T])$, define

$$Q(\Pi, w) = v_2(\Pi, w)^2$$

The quadratic variation of w (should it exist) is

$$Q(w) := \lim_{\|\Pi\| \to 0} Q(\Pi, w) \neq v_2(w)^2$$

Eg. $r < s < t$

$$\|w(t) - w(r)\|^2$$
$$= \|w(t) - w(s)\|^2 + \|w(s) - w(r)\|^2$$

Prop: If $(B_t)_{t \in [0,T]}$ is a $\mathbb{R}$-valued Brownian motion, and if $\Pi_m \in \mathcal{P}([0,T])$ with $\|\Pi_m\| \to 0$, then $Q(\Pi_m, B)$ converges in L^2 to T.
Moreover, if $\sum_m \|\Pi_m\| < \infty$, $Q(\Pi_m, B) \to T$ a.s. [HW]

Cor: Let $v_p(B)(\omega) := v_p((t \mapsto B_t(\omega))_{t \in [0,T]})$. If $p < 2$, $v_p(B) = \infty$ a.s.

Pf. Let $(\Pi_m)_{m \in \mathbb{N}}$ be a sequence in $\mathcal{P}([0,T])$ s.t. $\sum_m \|\Pi_m\| < \infty$

Let $\Omega_0 = \{Q(\Pi_m, B) \to T\}$

Suppose $\omega \in \Omega_0$ satisfies $v_p(B)(\omega) < \infty$. Then

$$Q(\Pi_m, B)(\omega) = v_2(\Pi_m, B_\cdot(\omega))^2$$

Now, $\|\Pi_m\| \to 0$, and $B_\cdot(\omega)$ is uniformly continuous.

Fix $\varepsilon > 0$, and let $\delta > 0$ be s.t. $|s-t| < \delta \Rightarrow |\omega(s) - \omega(t)| < \varepsilon$.

For all large m, $\|\Pi_m\| < \delta$, so

$$\max_{t_k \in \Pi_m} |B_{t_k}(\omega) - B_{t_{k-1}}(\omega)|^{2-p}$$

Thus $Q(\Pi_m, B)(\omega)$

(This is also true for $p = 2$; harder to prove.)

Cor: If $\alpha > \frac{1}{2}$, Brownian motion $(B_t)_{t \in [0,T]}$ is a.s. not C^α.

Pf. Let $(\Pi_m)_{m \in \mathbb{N}}$ and $\Omega_0 = \{ Q(\Pi_m, B) \to T \}$ as above, so $\mathbb{P}(\Omega_0) = 1$.

If $t \mapsto B_t(\omega)$ is C^α for some $\omega \in \Omega_0$,

$$Q(\Pi_m, B)(\omega) = \sum_{t_j \in \Pi_m} (B_{t_j}(\omega) - B_{t_{j-1}}(\omega))^2$$

So, with probability 1, Brownian motion is not C^α for any $\alpha > \frac{1}{2}$. (Again, this is also true for $\alpha = \frac{1}{2}$.) That's true on any interval $[0,T]$. In fact, it's even true locally.