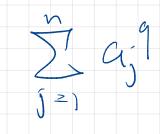


If V, (w) < 00, we say whas bounded variation

It is possible for a path to have unbounded variation, but still $v_p(w) < \infty$ for some p > 1. But if $v_p(w) < \infty$, $v_q(w) < \infty$ for q > p. Prop For we C(E0,T1,S), $p \mapsto v_p(w)$ is a decreasing function.

Pf. Follows from $p \mapsto ov_p(TT, w)$ being decreasing for any TT. For that, just note that for any $(a_j)_{j=1}^n \ge 0$

end isp<q<00



In the p=1 case, there is an alternative way to compute V(w)

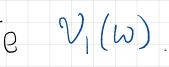
 $V_{1}(w) = \sup_{T \in \mathcal{P}([0,T])} V_{1}(TT, w)$

The reason is the following:

Lemma If $TT \subseteq TT'$ then $V_1(TT_1, w) \leq V_1(TT'_1, w)$.

 $Pf. TT = \{ c = t_0 < t_1 < t_2 < -- < t_n \}$

The same is not the for p>1. Indeed, it is possible for lime vp(TT, w) < as but vp(w)=co. BV (2,< co) paths are provisely the Riemann-Stieltjes integrators: $\int f dw = \lim_{t \to 0} \sum_{t \in T} f(t_{j}^{*}) (w(t_{j}) - w(t_{j-1}))$



Quadratic Variation For we $C(\Omega, T)$, $T \in \mathcal{P}(\Omega, T)$, define $<math>Q(T, w) = \mathcal{V}_2(TT, w)^2$

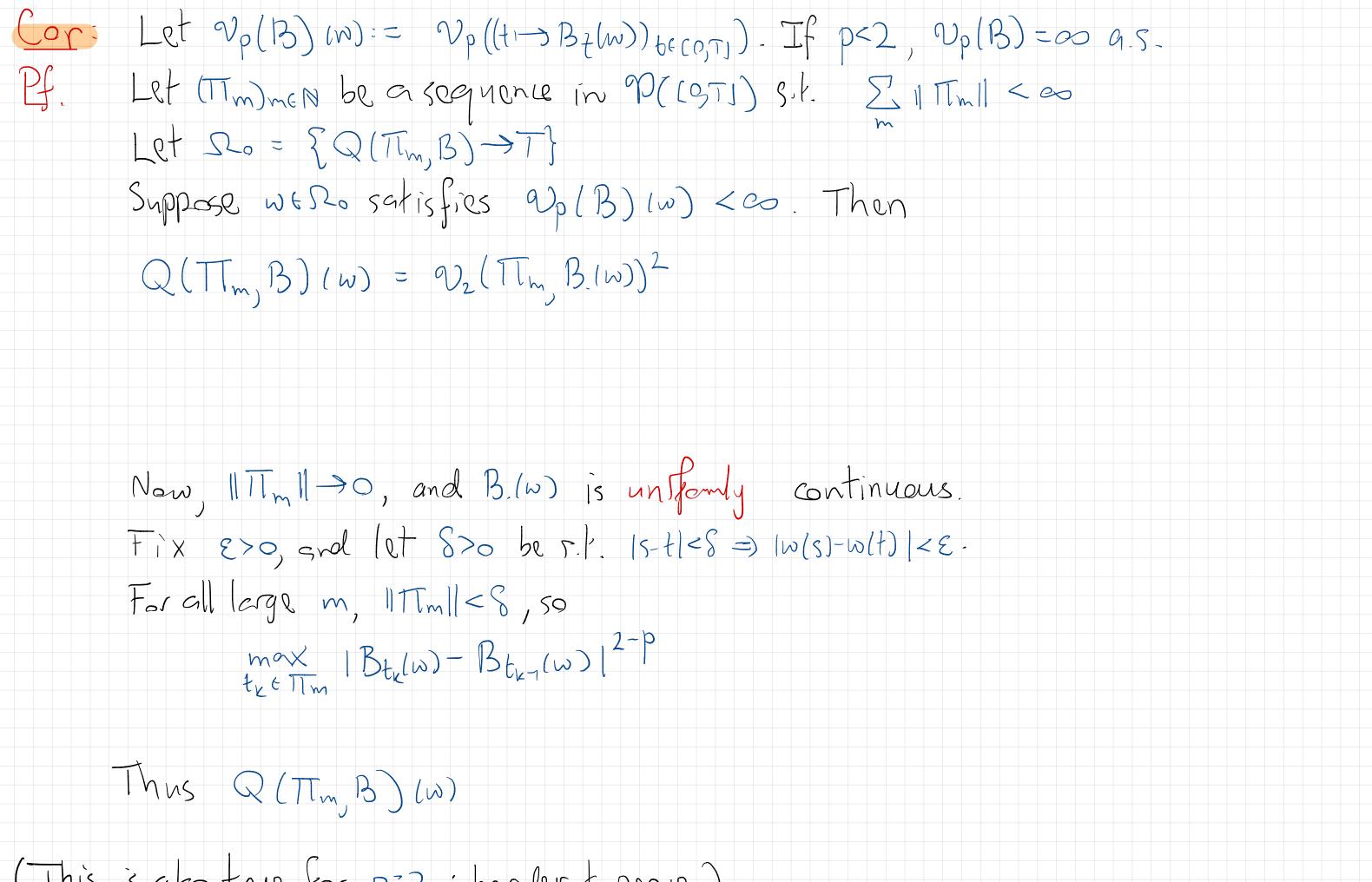
The quadratic variation of w (should it exist) is

 $Q(\omega) = \lim_{\|TT\| \to 0} Q(TT, \omega) \neq N_2(\omega)^2$

Eq. r < s < t $\|w(t) - w(r)\|^2$

= $\|w(t) - w(s)\|^2 + \|w(s) - w(r)\|^2$

Prop: If (Bt) teco, TJ is a R-valued Brownian motion, and if TIME @(CO, TJ) with IITIMII > 0, then Q(TIM, B) Converges in L² to T. Moreover, if ZIITIMII < co, Q(TIM, B) > T Q.S. [HW]



(This is also true for p=2; harder to prove.)

Cor If a>2, Brownian motion (Bf)tecory is as not Ca Pf. Let (TTm)meN and So= {Q(TTm, B) → T} as above, so P(So)=1. If tim By(w) is CX for some wesh, $Q(TTm, B)(w) = \sum_{t, \in TTm}^{T} (B_{t, j}(w) - B_{t, j}(w))^2$ So, with probability 1, Brownian motion is not Ca for any $\alpha > \frac{1}{2}$. (Again, this is also the for $\alpha = \frac{1}{2}$.) That's Free on any interval CoTJ. In fact, it's even the Jocally.

