Regularity of Paths

Let $(S, \| \|)$ be a normed space, and take $\omega \in C([0,T], S)$.
In general, let Π denote an interval partition of $[0,T]$

$$\Pi = \{0 = t_0 < t_1 < \cdots < t_n = T\} \in \mathcal{P}([0,T])$$

Def: Given $1 \le p < \infty$ and $\Pi \in \mathcal{P}([0,T])$, define

$$\mathcal{V}_p(\Pi, \omega) := \left(\sum_{j=1}^{n} \| \omega(t_j) - \omega(t_{j-1}) \|^p \right)^{1/p}$$

The p-variation $\mathcal{V}_p(\omega)$ of the path is $\mathcal{V}_p(\omega) := \sup_{\Pi \in \mathcal{P}([0,T])} \mathcal{V}_p(\Pi, \omega)$.

E.g. Suppose $\omega \in Lip([0,T])$. Then

$$\mathcal{V}_1(\Pi, \omega) = \sum_{j=1}^{n} \| \omega(t_j) - \omega(t_{j-1}) \| \le \|\omega\|_{Lip} \sum_{j=1}^{n} (t_j - t_{j-1})$$

$\|\omega(t) - \omega(s)\| \le \|\omega\|_{Lip} |t-s|$

$$= T \cdot \|\omega\|_{Lip} < \infty.$$

$\therefore \mathcal{V}_1(\omega) \le T \|\omega\|_{Lip} < \infty$. $\quad Lip([0,T], S) \subsetneq BV([0,T], S)$

If $\mathcal{V}_1(\omega) < \infty$, we say ω has bounded variation.

E.g. $f \in C^1$ $\quad \mathcal{V}_1(f) = \int_0^T |f'(t)| dt$. $\quad \|f\|_{Lip} = \max_{0 \le t \le T} |f'(t)|$.

It is possible for a path to have unbounded variation, but still $v_p(w) < \infty$ for some $p>1$. But if $v_p(w)<\infty$, $v_q(w)<\infty$ for $q>p$.

Prop For $w \in C([0,T],S)$, $p \mapsto v_p(w)$ is a decreasing function.

Pf. Follows from $p \mapsto v_p(\Pi, w)$ being decreasing for any Π.

For that, just note that for any $(a_j)_{j=1}^n \geq 0$ $\quad a_j = \| w(t_j) - w(t_{j-1}) \|$

and $1 \leq p < q < \infty$ $\quad q = p+r, \quad r>0$.

$$\sum_{j=1}^n a_j^q = \sum_{j=1}^n a_j^p a_j^r \leq \left(\max_i a_i\right)^r \sum_{j=1}^n a_j^p$$

$$\max_i a_i = \left(\max_i a_i^p\right)^{1/p}$$

$$= \left(\max_i a_i^p\right)^{r/p} \sum_{j=1}^n a_j^p \leq \left(\sum_{i=1}^n a_i^p\right)^{r/p} \sum_{j=1}^n a_j^p$$

$$= \left(\sum_{j=1}^n a_j^p\right)^{1+r/p}$$

$$1 + \frac{r}{p} = \frac{p+r}{p} = \frac{q}{p}$$

$$= \left(\sum_{j=1}^n a_j^p\right)^{q/p}.$$

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In the $p=1$ case, there is an alternative way to compute $v_1(\omega)$.

$$v_1(\omega) = \sup_{\Pi \in \mathcal{P}([0,T])} v_1(\Pi,\omega) = \lim_{n\to\infty} v_1(\Pi_n,\omega)$$

for any seq. $\Pi_n \in \mathcal{P}([0,T])$ s.t. $\Pi_n \subseteq \Pi_{n+1}$ & $\|\Pi_n\| = \max_j |t_j - t_{j-1}| \to 0$.

The reason is the following:

Lemma: If $\Pi \subseteq \Pi'$ then $v_1(\Pi,\omega) \le v_1(\Pi',\omega)$.

Pf. $\Pi = \{0 = t_0 < t_1 < t_2 < \cdots < t_n\}$

$\Pi' = \{0 = t_0 < t_1 < t_1' < t_2 < \cdots < t_n\}$

$$v_1(\Pi,\omega) = \|\omega(t_1)-\omega(t_0)\| + \|\omega(t_2)-\omega(t_1)\| + \cdots + \|\omega(t_n)-\omega(t_{n-1})\|$$

$$\le \|\omega(t_2)-\omega(t_1')\| + \|\omega(t_1')-\omega(t_1)\| = v_1(\Pi',\omega) \quad ///$$

The same is not true for $p>1$.

Indeed, it is possible for $\lim_{n\to\infty} v_p(\Pi_n,\omega) < \infty$ but $v_p(\omega) = \infty$.

BV ($v_1 < \infty$) paths are precisely the Riemann-Stieltjes integrators:

$t_j^* \in [t_{j-1}, t_j]$

$$\int f\,d\omega = \lim_{\|\Pi\|\to 0} \sum_{t_j \in \Pi} f(t_j^*)\,(\omega(t_j) - \omega(t_{j-1})).$$

Quadratic Variation

For $w \in C([0,T],S)$, $\Pi \in \mathcal{P}([0,T])$, define

$$Q(\Pi, w) = v_2(\Pi, w)^2 = \sum_{t_j \in \Pi} \|w(t_j) - w(t_{j-1})\|^2$$

The quadratic variation of w (should it exist) is

$$Q(w) := \lim_{\|\Pi\| \to 0} Q(\Pi, w) \neq v_2(w)^2 = \sup_{\Pi} v_2(\Pi, w)^2$$

Eg. $r < s < t$

$$\|w(t) - w(r)\|^2 = \|w(t) - w(s) + w(s) - w(r)\|^2$$

$$= \|w(t) - w(s)\|^2 + \|w(s) - w(r)\|^2 + 2\langle w(t) - w(s), w(s) - w(r)\rangle$$

could be ≥ 0, could be < 0.

Prop: If $(B_t)_{t \in [0,T]}$ is a $\mathbb{R}$-valued Brownian motion, and if $\Pi_m \in \mathcal{P}([0,T])$ with $\|\Pi_m\| \to 0$, then $Q(\Pi_m, B)(\omega) = Q(\Pi_m, t \mapsto B_t(\omega))$ converges in L^2 to T.

Moreover, if $\sum_m \|\Pi_m\| < \infty$, $Q(\Pi_m, B) \to T$ a.s. [HW]

Cor: Let $v_p(B)(\omega) := v_p((t \mapsto B_t(\omega))_{t\in[0,T]})$. If $p<2$, $v_p(B)=\infty$ a.s.

Pf. Let $(\Pi_m)_{m\in\mathbb{N}}$ be a sequence in $\mathcal{P}([0,T])$ s.t. $\sum_m \|\Pi_m\| < \infty$ $\therefore \|\Pi_m\| \to 0$.

Let $\Omega_0 = \{Q(\Pi_m, B) \to T\}$ $\therefore$ $P(\Omega_0)=1$.

Suppose $\omega\in\Omega_0$ satisfies $v_p(B)(\omega) < \infty$. Then

$$Q(\Pi_m, B)(\omega) = v_2(\Pi_m, B_\cdot(\omega))^2$$

$$= \sum_{t_j\in\Pi_m} |\underbrace{B_{t_j}(\omega) - B_{t_{j-1}}(\omega)}_{\Delta_j\omega}|^2 = \sum_{t_j\in\Pi_m} |\Delta_j\omega|^{2-p}\,|\Delta_j\omega|^p$$

$$\le \max_{t_i\in\Pi_m} |\Delta_i\omega|^{2-p} \underbrace{\sum_{t_j\in\Pi_m} |\Delta_j\omega|^p}_{= v_p(\Pi_m, B)^p \,\le\, v_p(B_\cdot)^p \,<\, \infty}$$

Now, $\|\Pi_m\|\to 0$, and $B_\cdot(\omega)$ is uniformly continuous.

Fix $\varepsilon>0$, and let $\delta>0$ be s.t. $|s-t|<\delta \Rightarrow |\omega(s)-\omega(t)|<\varepsilon$.

For all large m, $\|\Pi_m\|<\delta$, so

$$\max_{t_k\in\Pi_m} |B_{t_k}(\omega) - B_{t_{k-1}}(\omega)|^{2-p} < \varepsilon^{2-p}.$$

$$|t_k - t_{k-1}| \le \|\Pi_m\| < \delta$$

Thus $Q(\Pi_m, B)(\omega) \to 0$. ↯
$\to T \neq 0$.

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(This is also true for $p=2$; harder to prove.)

Cor: If $\alpha > \frac{1}{2}$, Brownian motion $(B_t)_{t\in[0,T]}$ is a.s. not C^α.

Pf. Let $(\Pi_m)_{m\in\mathbb{N}}$ and $\Omega_0 = \{Q(\Pi_m, B) \to T\}$ as above, so $\mathbb{P}(\Omega_0) = 1$.
If $t \mapsto B_t(\omega)$ is C^α for some $\omega \in \Omega_0$, $|B_t(\omega) - B_s(\omega)| \leq K(\omega)\,|s-t|^\alpha$.

$$Q(\Pi_m, B)(\omega) = \sum_{t_j \in \Pi_m} (B_{t_j}(\omega) - B_{t_{j-1}}(\omega))^2$$

$$\leq K(\omega)^2 \sum_{t_j \in \Pi_m} |t_j - t_{j-1}|^{(2\alpha-1)} |t_j - t_{j-1}|^1$$

$$\leq K(\omega)^2 \max_i |t_i - t_{i-1}|^{2\alpha-1} \underbrace{\sum_j (t_j - t_{j-1})}_{T}$$

$$\leq K(\omega)^2 T \, \|\Pi_m\|^{2\alpha-1} \longrightarrow 0, \quad m \to \infty$$

$$Q(\Pi_m, B)(\omega) \to T \quad ///$$

So, with probability 1, Brownian motion is not C^α for any $\alpha > \frac{1}{2}$. (Again, this is also true for $\alpha = \frac{1}{2}$.)
That's true on any interval $[0,T]$. In fact, it's even true locally.