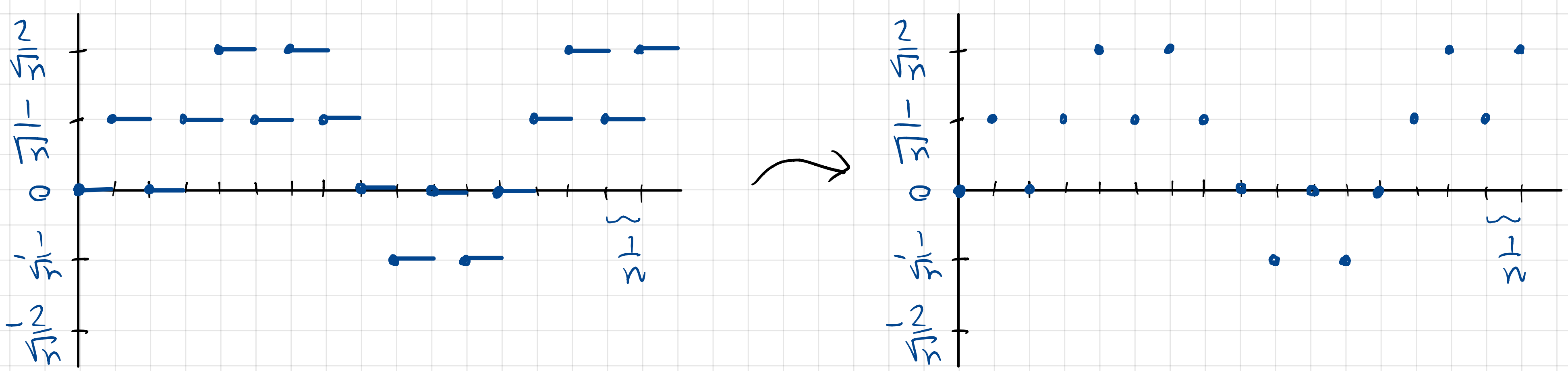
Last time, we proved the following:

If $\{X_n\}_{n=1}^{\infty}$ are iid L^2 rv's with $\mathbb{E}[X_n]=0$, $\mathbb{E}[X_n^2]=1$, $S_n = \sum_{k=1}^{n} X_k$ and $W_n(t) := \frac{1}{\sqrt{n}} S_{\lfloor nt \rfloor}$, then $W_n \xrightarrow{f.d.} B$ ← Brownian motion.

We'd like to upgrade this result from convergence in f.d. distributions to weak convergence...

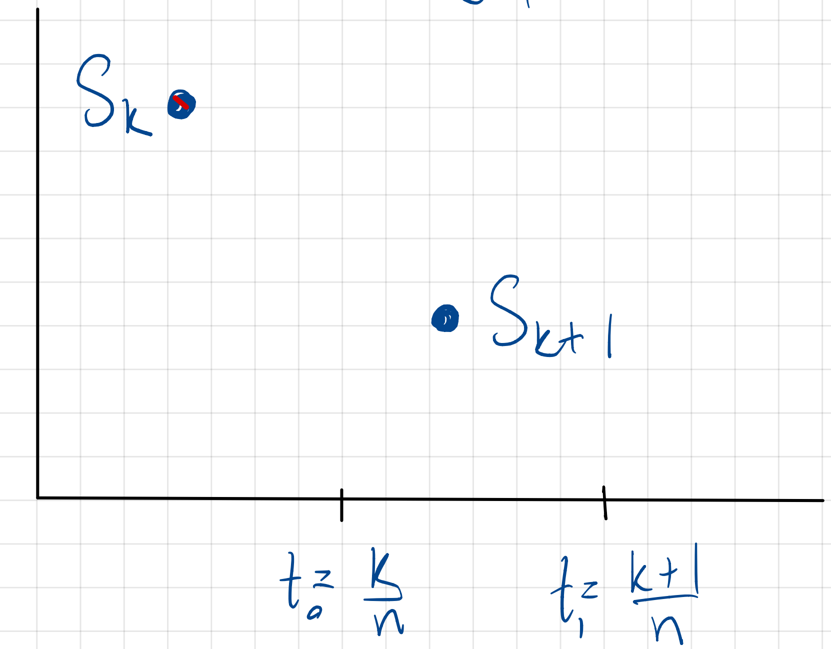
There's a quick fix:



W_n

Def: Let $\{X_n\}_{n=1}^{\infty}$ be iid L^2 rv's with $E[X_n]=0$, $E[X_n^2]=1$. Set $S_n=\sum_{k=1}^{n} X_k$.

Define $B_n(t) := \frac{1}{\sqrt{n}}\left(S_{\lfloor nt\rfloor} + (nt-\lfloor nt\rfloor)X_{\lfloor nt\rfloor+1}\right)$



$\therefore$ B_n is a continuous process.

And: it is very close to W_n.

Prop: $B_n \xrightarrow{f.d.} B$ Brownian motion.

Pf. Let $t_1,\dots,t_k \geq 0$. Set $X_n = (B_n(t_1),\dots,B_n(t_k))$
$Y_n = (W_n(t_1),\dots,W_n(t_k))$ $\in \mathbb{R}^k$

We've proved that $Y_n \to_w (B(t_1),\dots,B(t_k))$

we'll now show that $\Delta_n = X_n - Y_n \to_p 0$; by Slutzky's thm, it then follows that

$$\Delta_n = X_n - Y_n = (B_n(t_1), \dots, B_n(t_k)) - (W_n(t_1), \dots, W_n(t_k))$$

$$= \frac{1}{\sqrt{n}} \left((nt_1 - \lfloor nt_1 \rfloor) X_{\lfloor nt_1 \rfloor + 1}, \dots, (nt_k - \lfloor nt_k \rfloor) X_{\lfloor nt_k \rfloor + 1} \right)$$

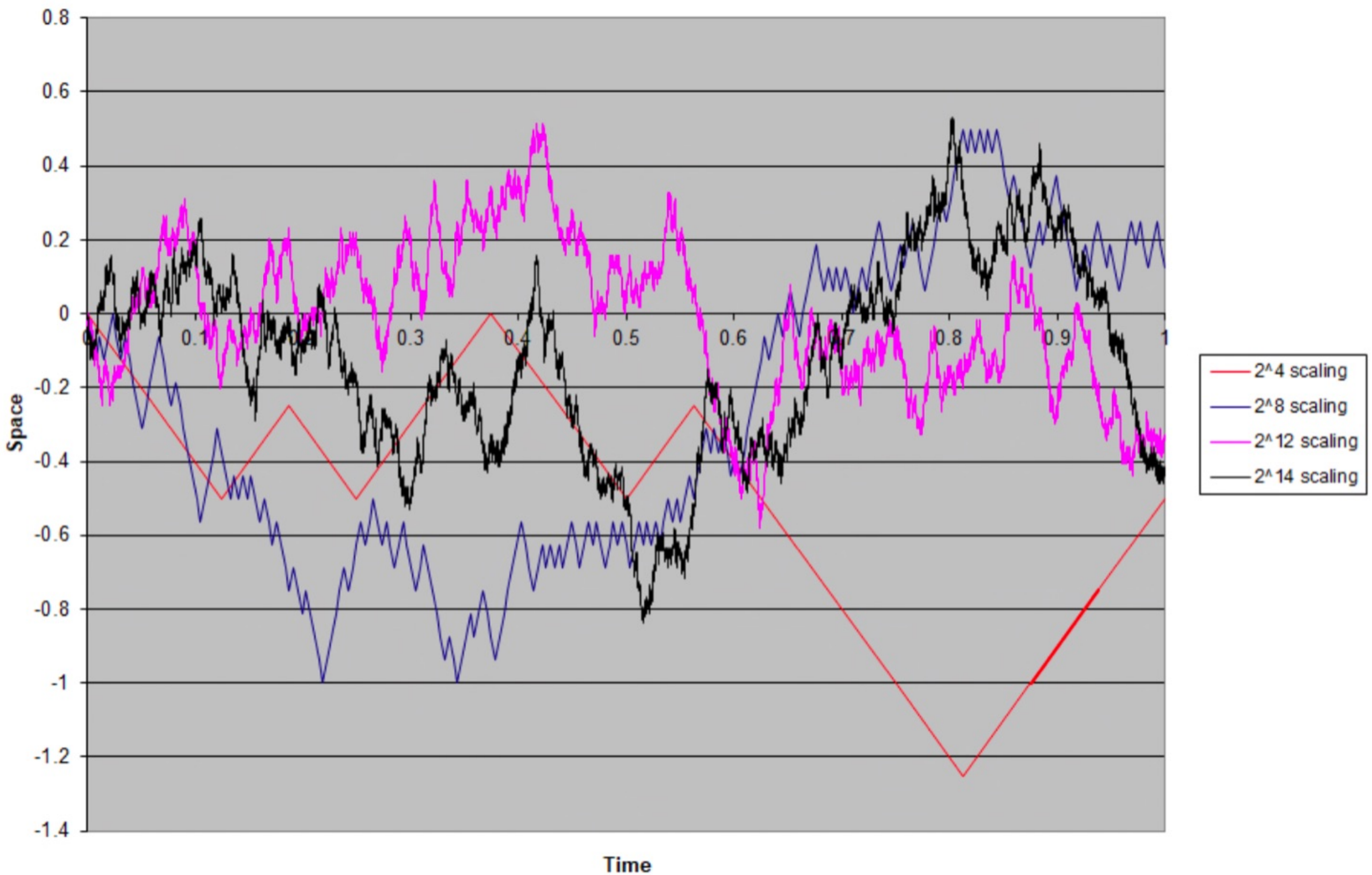
$$\mathbb{P}(\|\Delta_n\| > \varepsilon) = \mathbb{P}\left(\sum_{j=1}^{k} \left| \frac{1}{\sqrt{n}} (nt_j - \lfloor nt_j \rfloor) X_{\lfloor nt_j \rfloor + 1} \right|^2 > \varepsilon^2 \right)$$

So: now we have processes $B_n \in C(\mathbb{R}_+, \mathbb{R})$,
the same (path) state space as Brownian motion B,
and $B_n \xrightarrow{f.d.} B$.

Theorem: (Donsker's Functional CLT / Donsker's Invariance Principle)

$$\forall T > 0, \quad B_n|_{[0,T]} \to_w B|_{[0,T]}$$

I.e. $\forall F \in C_b(C([0,T], \mathbb{R}))$, $\mathbb{E}[F(B_n)] \xrightarrow{n\to\infty} \mathbb{E}[F(B)]$.



Pf. We know $B_n \xrightarrow{f.d.} B$, and all are continuous processes.
Therefore, to conclude $B_n|_{[0,T]} \to_w B|_{[0,T]}$, it suffices (by [Lec 52.3]) to show that $\{B_n\}_{n\in\mathbb{N}}$ is tight.

To establish tightness, it suffices (by [Lec 52.2]) to show that $\{B_n\}_{n\in\mathbb{N}}$ satisfy Kolmogorov's tightness criteria:

$$\sup_n \mathbb{E}[\,|B_n(s)-B_n(t)|^p\,] \le C|s-t|^{1+\varepsilon} \quad \forall s,t\ge 0 \text{ for some } C,\varepsilon>0,\ p\ge 1+\varepsilon$$

and $\{B_n(0)\}_{n\in\mathbb{N}}$ is tight.

This is true with the problem as stated.
But to give a palatable proof, we're going to make the slightly stronger assumption

$$\boxed{X_n \in L^4}$$

Claim: Under this assumption,

$$\forall n \quad \mathbb{E}[\,|B_n(s)-B_n(t)|^4\,] \le C\,|s-t|^2$$

$$B_n(t) := \frac{1}{\sqrt{n}}\left(S_{\lfloor nt \rfloor} + (nt - \lfloor nt \rfloor) X_{\lfloor nt \rfloor + 1}\right)$$

To estimate $\mathbb{E}[|B_n(s) - B_n(t)|^4]$, we consider three regimes for $|s-t|$:

"Very Close" "Close" "Far"

- $l = k$. $B_n(t) - B_n(s) =$

$$\therefore \|B_n(t) - B_n(s)\|_4 = \sqrt{n}(t-s)\|X_k\|_4$$

- $l = k+1$. $B_n(t) - B_n(s) = \frac{1}{\sqrt{n}}\left(\left(S_k + (nt-k)X_{k+1}\right) - \left(S_{k-1} + (ns-(k-1))\right)X_k\right)$

$$\therefore \|B_n(t) - B_n(s)\|_4 \leq \frac{1}{\sqrt{n}}\left(|nt-k|\,\|X_{k+1}\|_4 + |k-ns|\,\|X_k\|_4\right)$$

- $l \geq k+2$ $\quad s \in [\frac{k-1}{n}, \frac{k}{n})$, $t \in [\frac{l-1}{n}, \frac{l}{n})$

$$\|B_n(t) - B_n(s)\|_4 \leq \|B_n(t) - B_n(\tfrac{l-1}{n})\|_4 + \|B_n(\tfrac{l-1}{n}) - B_n(\tfrac{k-1}{n})\|_4 + \|B_n(\tfrac{k-1}{n}) - B_n(s)\|_4$$

$\frac{1}{\sqrt{n}}(S_{l-1} - S_{k-1}) \overset{d}{=} \frac{1}{\sqrt{n}} S_{l-k}$ So we need to estimate $\|S_m\|_4$.

Lemma: $\mathbb{E}[S_m^4] = m\mathbb{E}[X_1^4] + 3m(m-1)$

Pf. $\mathbb{E}[(S_{m-1} + X_m)^4] = \mathbb{E}[S_{m-1}^4] + 4\mathbb{E}[S_{m-1}^3 X_m + S_{m-1} X_m^3] + 6\mathbb{E}[S_{m-1}^2 X_m^2] + \mathbb{E}[X_m^4]$

In particular, $\mathbb{E}[S_m^4] \leq m\mathbb{E}[X_1^4] + 3m(m-1)\mathbb{E}[X_1^4]$

$\leq 3m^2 \mathbb{E}[X_1^4]$.

$\therefore \|B_n(t) - B_n(s)\|_4 \leq \left(\sqrt{t - \frac{l-1}{n}} + \sqrt{s - \frac{k-1}{n}} + \frac{1}{\sqrt{n}} \cdot 2\sqrt{l-k}\right)\|X_1\|_4$

Thus, all together, we find that for any $s<t$,

$$\mathbb{E}[\,|B_n(s)-B_n(t)|^4\,] \leq C\,(s-t)^2$$

where $C = \max\{1, 4, (\sqrt{2}+\sqrt{6})^4\}\,\mathbb{E}[X_1^4]$

This (together with the fact that $B_n(0)=0 \;\forall n$ so forms a tight sequence) proves Kolmogorov's tightness criteria, and so concludes the proof that $B_n \rightarrow_w B$.