Last time, we proved the following:

If $\{X_n\}_{n=1}^{\infty}$ are iid L^2 rv's with $\mathbb{E}[X_n]=0$, $\mathbb{E}[X_n^2]=1$, $S_n = \sum_{k=1}^{n} X_k$ and $W_n(t) := \frac{1}{\sqrt{n}} S_{\lfloor nt \rfloor}$, then $W_n \xrightarrow[f.d.]{} B$ ← Brownian motion.

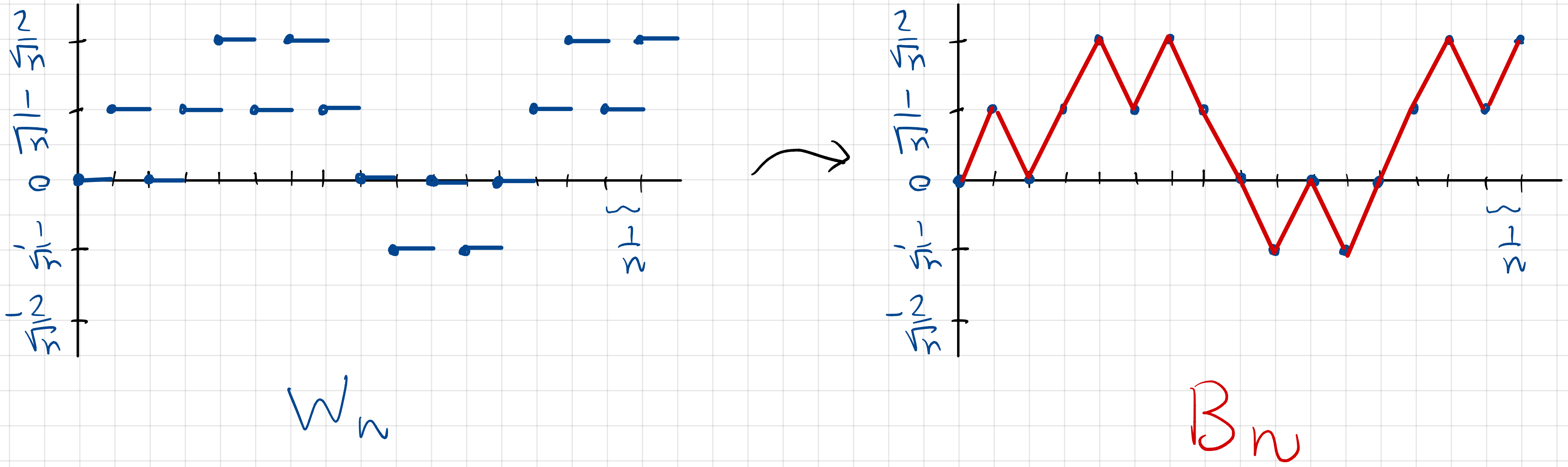
We'd like to upgrade this result from convergence in f.d. distributions to weak convergence...

$\mathrm{Law}(W_n) \in \mathrm{Prob}(\mathbb{R}^{\mathbb{R}_+})$

$\notin \mathrm{Prob}(C(\mathbb{R}_+, \mathbb{R}))$

$\mathrm{Law}(B) \in \mathrm{Prob}(C(\mathbb{R}_+, \mathbb{R}))$

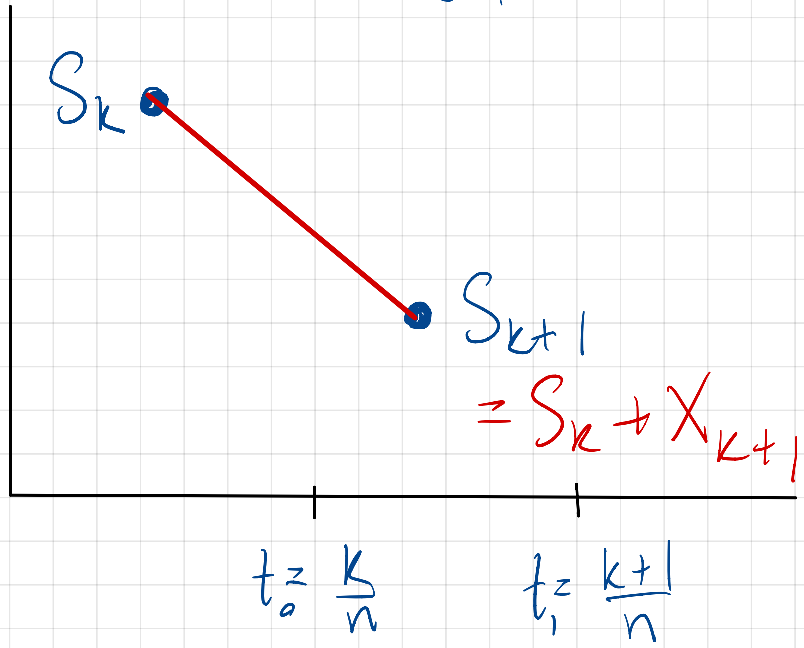
There's a quick fix:



Def: Let $\{X_n\}_{n=1}^{\infty}$ be iid L^2 rv's with $E[X_n]=0$, $E[X_n^2]=1$. Set $S_n=\sum_{k=1}^{n} X_k$.

Define $B_n(t) := \frac{1}{\sqrt{n}}\left(S_{\lfloor nt \rfloor} + (nt - \lfloor nt \rfloor) X_{\lfloor nt \rfloor + 1}\right)$

$$= W_n(t) + \underbrace{\frac{((nt))}{\sqrt{n}}}_{=0 \text{ if } nt\in\mathbb{Z}} X_{\lfloor nt \rfloor +1}.$$



$$\lim_{t \uparrow \frac{k+1}{n}} B_n(t) = \frac{1}{\sqrt{n}}\left(S_k + 1 \cdot X_{k+1}\right)$$
$$= W_n\left(\frac{k+1}{n}\right) = B_n\left(\frac{k+1}{n}\right).$$

∴ B_n is a continuous process.

And: it is very close to W_n.

Prop: $B_n \xrightarrow{f.d.} B$ Brownian motion.

Pf. Let $t_1, \dots, t_k \geq 0$. Set $X_n = (B_n(t_1), \dots, B_n(t_k))$
$Y_n = (W_n(t_1), \dots, W_n(t_k))$ $\in \mathbb{R}^k$

We've proved that $Y_n \xrightarrow{w} (B(t_1), \dots, B(t_k)). = Z$

we'll now show that $\Delta_n = X_n - Y_n \xrightarrow{p} 0$; by Slutzky's thm, it then follows that $X_n = Y_n + \Delta_n \xrightarrow{w} Z + 0$. ✓

$$\Delta_n = X_n - Y_n = (B_n(t_1), \dots, B_n(t_k)) - (W_n(t_1), \dots, W_n(t_k))$$

$$= \frac{1}{\sqrt{n}} \left((nt_1 - \lfloor nt_1 \rfloor) X_{\lfloor nt_1 \rfloor + 1}, \dots, (nt_k - \lfloor nt_k \rfloor) X_{\lfloor nt_k \rfloor + 1} \right)$$

$$\mathbb{P}(\|\Delta_n\| > \varepsilon) = P\left(\sum_{j=1}^{k} \left| \frac{1}{\sqrt{n}} (nt_j - \lfloor nt_j \rfloor) X_{\lfloor nt_j \rfloor + 1} \right|^2 > \varepsilon^2 \right)$$

$$\left\{ \sum_{j=1}^{k} |f_j|^2 > \varepsilon^2 \right\} \subseteq \left\{ \exists_j : |f_j|^2 > \frac{\varepsilon^2}{k} \right\}$$

$$\le \sum_{j=1}^{k} \mathbb{P}\Big(\underbrace{\frac{1}{\sqrt{n}} |(nt_j - \lfloor nt_j \rfloor) X_{\lfloor nt_j \rfloor + 1}| > \varepsilon}_{\text{Markov}} \Big)$$

$$\text{Markov} \quad \le \frac{\mathbb{E}\left[\frac{1}{\sqrt{n}} |((nt_j)) X_{\lfloor nt_j \rfloor + 1}| \right]}{\varepsilon}$$

$$\le \frac{1}{\sqrt{n}} \cdot \frac{1}{\varepsilon} \mathbb{E}[|X_1|]$$

$$\le \frac{k}{\sqrt{n}} \frac{1}{\varepsilon} \mathbb{E}[|X_1|] \longrightarrow 0 \quad \text{as } n \to \infty. \quad ///$$

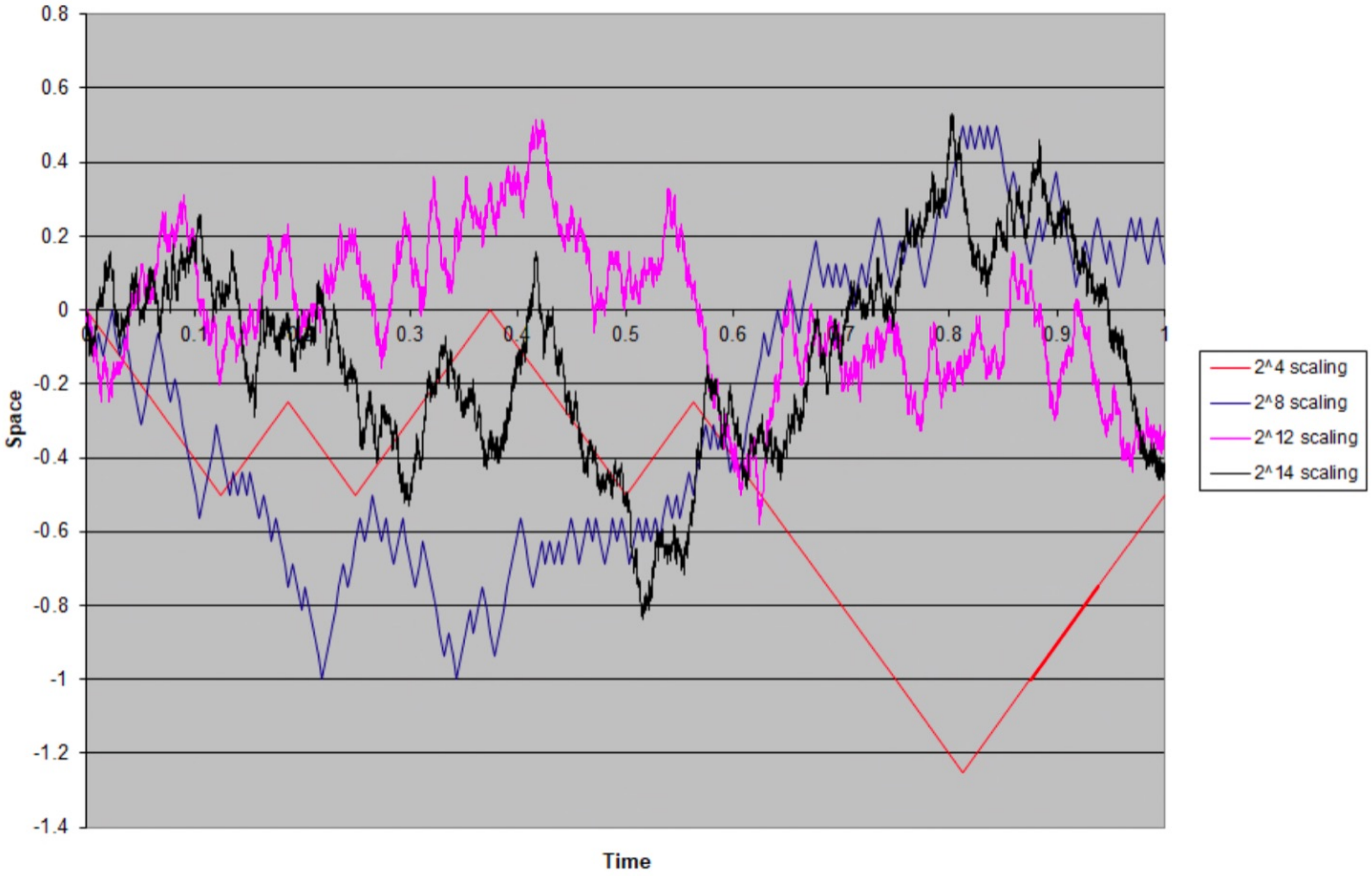
So: now we have processes $B_n \in C(\mathbb{R}_+, \mathbb{R})$, the same (path) state space as Brownian motion B, and $B_n \xrightarrow{f.d.} B$.

$\searrow$ upgrade to $B_n \xrightarrow{w} B$?

Theorem: (Donsker's Functional CLT / Donsker's Invariance Principle)

$$\forall T > 0, \qquad B_n|_{[0,T]} \xrightarrow{w} B|_{[0,T]}$$

I.e. $\forall F \in C_b(C([0,T],\mathbb{R}))$, $\mathbb{E}[F(B_n)] \xrightarrow{n\to\infty} \mathbb{E}[F(B)]$.



Pf. We know $B_n \xrightarrow{f.d.} B$, and all are continuous processes. Therefore, to conclude $B_n|_{[0,T]} \rightarrow_w B|_{[0,T]}$, it suffices (by [Lec 52.3]) to show that $\{B_n\}_{n\in\mathbb{N}}$ is tight.

To establish tightness, it suffices (by [Lec 52.2]) to show that $\{B_n\}_{n\in\mathbb{N}}$ satisfy Kolmogorov's tightness criteria:

$$\sup_n \mathbb{E}[\,|B_n(s)-B_n(t)|^p\,] \leq C|s-t|^{1+\varepsilon} \quad \forall s,t \geq 0 \text{ for some } C,\varepsilon > 0,\ p \geq 1+\varepsilon$$

and $\{B_n(0)\}_{n\in\mathbb{N}}$ is tight. $B_n(0) \equiv 0\ \forall n.$

This is true with the problem as stated. But to give a palatable proof, we're going to make the slightly stronger assumption

$$\boxed{X_n \in L^4}$$

Claim: Under this assumption,

$$\forall n \quad \mathbb{E}[\,|B_n(s)-B_n(t)|^4\,] \leq \underset{\uparrow\text{unif. in } n.}{C}\,|s-t|^2$$

$$\mathbb{R}_+ = \bigcup_{k=1}^{\infty} \Big[\underset{\overset{\uparrow}{t}}{\tfrac{k-1}{n}, \tfrac{k}{n}} \Big) \qquad B_n(t) := \frac{1}{\sqrt{n}} \left(S_{\lfloor nt \rfloor} + (nt - \lfloor nt \rfloor) X_{\lfloor nt \rfloor + 1} \right)$$

$$= \frac{1}{\sqrt{n}} \left(S_{k-1} + (nt - (k-1)) X_k \right)$$

To estimate $\mathbb{E}[|B_n(s) - B_n(t)|^4]$, we consider three regimes for $|s-t|$:

$s \in [\frac{k-1}{n}, \frac{k}{n})$, $t \in [\frac{l-1}{n}, \frac{l}{n})$ "Very Close" $l = k$ "Close" $l = k+1$ "Far" $l \geq k+2$.

- $l = k$. $B_n(t) - B_n(s) = \frac{1}{\sqrt{n}} (n(t-s)) X_k$.

$$\therefore \ \|B_n(t) - B_n(s)\|_4^4 = \left(\sqrt{n}(t-s) \|X_1\|_4 \right)^2 = \cancel{n}^2 (t-s)^4 \, \mathbb{E}[X_1^4] \leqslant \mathbb{E}[X_1^4](t-s)^2$$

$$\lll \frac{1}{\cancel{n}^2}(t-s)^2$$

- $l = k+1$. $B_n(t) - B_n(s) = \frac{1}{\sqrt{n}} \left(\underbrace{(S_k + (nt-k) X_{k+1}) - (S_{k-1}}_{X_k} + (ns - (k-1)) X_k \right)$

$ns < k \leq nt$

$$= \frac{1}{\sqrt{n}} \left((k - ns) X_k + (nt - k) X_{k+1} \right)$$

$$\therefore \ \|B_n(t) - B_n(s)\|_4 \leqslant \frac{1}{\sqrt{n}} \left(\overset{(nt-k)}{|nt-k|} \|X_{k+1}\|_4 + \overset{(k-ns)}{|k-ns|} \|X_k\|_4 \right)$$

$$= \frac{1}{\sqrt{n}} (nt - \cancel{k} + \cancel{k} - ns) \|X_1\|_4$$

$$= \sqrt{n}(t-s) \|X_1\|_4 .$$

$\hookrightarrow t - s \leqslant \frac{2}{n}$

$$\hookrightarrow \therefore \mathbb{E}[(B_n(t) - B_n(s))^4] \leq n^2 (t-s)^4 \, \mathbb{E}[X_1^4]$$

$$\leqslant 4 \mathbb{E}[X_1^4](t-s)^2$$

- $l \geq k+2$ $\quad s \in [\frac{k-1}{n}, \frac{k}{n})$, $t \in [\frac{l-1}{n}, \frac{l}{n})$ $\quad t-s \geq \frac{2}{n}$.

$$\|B_n(t)-B_n(s)\|_4 \leq \|B_n(t)-B_n(\tfrac{l-1}{n})\|_4 + \|B_n(\tfrac{l-1}{n})-B_n(\tfrac{k-1}{n})\|_4 + \|B_n(\tfrac{k-1}{n})-B_n(s)\|_4$$

$$\leq \sqrt{t-\frac{l-1}{n}}\,\|X_1\|_4 \qquad\qquad \leq \sqrt{s-\frac{k-1}{n}}\,\|X_1\|_4 .$$

$\frac{1}{\sqrt{n}}(S_{l-1}-S_{k-1}) \overset{d}{=} \frac{1}{\sqrt{n}} S_{l-k}$ So we need to estimate $\|S_m\|_4$.

Lemma: $E[S_m^4] = mE[X_1^4] + 3m(m-1)$ $\quad 6\cdot\sum_{j=1}^{m}(j-1)$.

Pf. $E[(S_{m-1}+X_m)^4] = E[S_{m-1}^4] + 4E[\underbrace{S_{m-1}^3X_m}_{0} + \underbrace{S_{m-1}X_m^3}_{0}] + 6E[S_{m-1}^2X_m^2] + E[X_m^4]$

$\quad // \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad E[S_{m-1}^2]E[X_m^2]$

$\quad \| \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad /// \qquad\quad m-1. \quad 1$

$E[S_{m-1}^4] + E[X_m^4] + 6(m-1)$.

In particular, $E[S_m^4] \leq mE[X_1^4] + 3m(m-1)E[X_1^4]$ $\quad \geq 1.$

$$\leq 3m^2 E[X_1^4].$$

$$\therefore \|B_n(t)-B_n(s)\|_4 \leq \Big(\underbrace{\sqrt{t-\tfrac{l-1}{n}}}_{\leq \frac{1}{n}} + \underbrace{\sqrt{s-\tfrac{k-1}{n}}}_{\leq \frac{1}{n}} + \frac{1}{\sqrt{n}}\cdot 2\sqrt{l-k}\Big)\|X_1\|_4$$

$\leq 2\sqrt{\frac{t-s}{2}}$

$\wedge$

$(\sqrt{2}+\sqrt{6})\sqrt{t-s}\,\|X_1\|_4$

$2\sqrt{\frac{l}{n}-\frac{k}{n}} = 2\sqrt{\underbrace{\frac{l-1}{n}-\frac{k}{n}}_{\leq t-s}+\underbrace{\frac{1}{n}}_{\leq\frac{1}{2}(t-s)}}$

$\leq \sqrt{6(t-s)}$

Thus, all together, we find that for any $s<t$,

$$\mathbb{E}[|B_n(s)-B_n(t)|^4] \leq C(s-t)^2$$

where $C = \max\{1, 4, (\sqrt{2}+\sqrt{6})^4\} \mathbb{E}[X_1^4]$

$$\doteq 223$$

This (together with the fact that $B_n(0)=0$ $\forall n$ so forms a tight sequence) proves Kolmogorov's tightness criteria, and so concludes the proof that $B_n \to_w B$.

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