Recall the (basic) Central Limit Theorem:

If $(X_n)_{n=1}^{\infty}$ are i.i.d. L^2 random variables with $E[X_n]=0$ and $E[X_n^2]=1$, then (with $S_n = X_1 + \cdots + X_n$)

$$\frac{S_n}{\sqrt{n}} \to_w \mathcal{N}(0,1).$$

$(S_n)_{n=1}^{\infty}$ is a discrete-time stochastic process; if $X_n \overset{d}{=} \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}$, it is SRW.

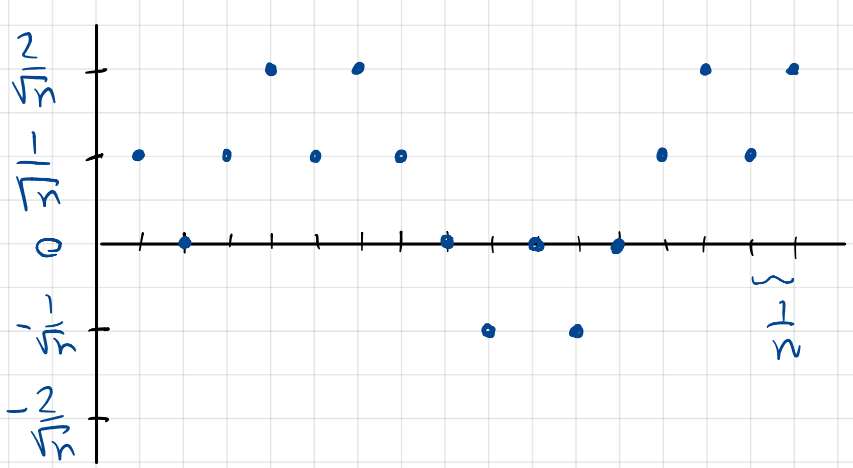
Can we make a continuous time process out of it?

Rough idea: just rescale $W_n(t)$ "=" $\frac{S_{nt}}{\sqrt{n}}$

Doesn't make sense if $t \notin \mathbb{N}$ in general.
But we can fix that.

Def: Given $\{X_n\}_{n=1}^{\infty}$ as above, define for $t \geq 0$

$$W_n(t) := \frac{1}{\sqrt{n}} S_{\lfloor nt \rfloor}$$



vertical (space) scaled $\frac{1}{\sqrt{n}}$
horizontal (time) scaled $\frac{1}{\sqrt{n}}$

Prop: The processes $(W_n(t))_{t\geq 0}$ converge to Brownian motion $(B(t))_{t\geq 0}$ as $n\to\infty$, in finite-dimensional distributions.

Pf. We will work with increments. Fix $t > s \geq 0$.

$$W_n(t) - W_n(s) = \frac{1}{\sqrt{n}}\left(S_{\lfloor nt\rfloor} - S_{\lfloor ns\rfloor}\right)$$

By the CLT $\underbrace{\frac{1}{\sqrt{\lfloor nt\rfloor - \lfloor ns\rfloor}} \cdot \sum_{\lfloor ns\rfloor < k \leq \lfloor nt\rfloor} X_k}_{Z_n(s,t)} \to_w \mathcal{N}(0,1)$

$$\frac{\lfloor nt\rfloor}{n} =$$

So, the increments of $(W_n(t))_{t\geq 0}$ are asymptotically normally distributed:

For $0 \leq s < t$, $W_n(t) - W_n(s) \xrightarrow[w]{n\to\infty} \mathcal{N}(0, t-s)$

Moreover, note that $W_n(t) - W_n(s) = \frac{1}{\sqrt{n}} \sum_{\lfloor ns\rfloor < k \leq \lfloor nt\rfloor} X_k$

Now, let $0 = t_0 < t_1 < \cdots < t_k$.

We know $(W_n(t_j) - W_n(t_{j-1}))_{j=1}^{k}$ are independent & $(B(t_j) - B(t_{j-1}))_{j=1}^{k}$ are independent.

On the last slide, we proved that

$$W_n(t_j) - W_n(t_{j-1}) \xrightarrow[w]{n\to\infty} B(t_j) - B(t_{j-1}) \quad j = 1..k$$

Thus, by the last proposition in [Lec. 53.1],

$$(W_n(t_1) - W_n(t_0), \ldots, W_n(t_k) - W_n(t_{k-1}))$$

$$\xrightarrow[w]{n\to\infty} (B(t_1) - B(t_0), \ldots, B(t_k) - B(t_{k-1})).$$

$$(W_n(t_1)-W_n(t_0), \ldots, W_n(t_k)-W_n(t_{k-1})) \xrightarrow[w]{n\to\infty} (B(t_1)-B(t_0), \ldots, B(t_k)-B(t_{k-1})).$$

Consider the function $f: \mathbb{R}^k \to \mathbb{R}^k$

By the continuous mapping theorem,

$$f(W_n(t_1)-W_n(t_0), \ldots, W_n(t_k)-W_n(t_{k-1})) \xrightarrow[w]{n\to\infty} f(B(t_1)-B(t_0), \ldots, B(t_k)-B(t_{k-1}))$$

Finally, if the $\{t_j\}_{j=1}^{k}$ are not indexed in increasing order, permute them and permute the variables of f accordingly, at the beginning and at the end.

So, $W_n \xrightarrow{f.d.} B$. What about $W_n \to_w B$?