Recall the (basic) Central Limit Theorem:

If $(X_n)_{n=1}^{\infty}$ are i.i.d. L^2 random variables with $E[X_n]=0$ and $E[X_n^2]=1$, then (with $S_n = X_1+\cdots+X_n$)

$$\frac{S_n}{\sqrt{n}} \to_w \mathcal{N}(0,1).$$

$(S_n)_{n=1}^{\infty}$ is a discrete-time stochastic process; if $X_n \overset{d}{=} \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}$, it is SRW.

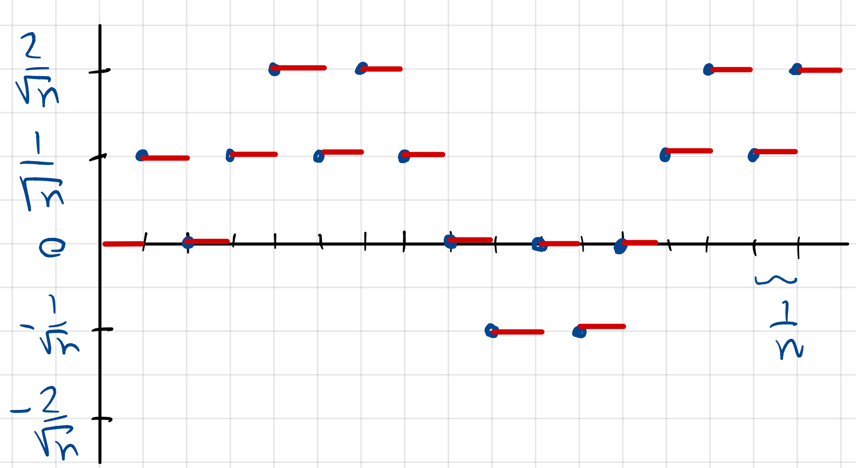
Can we make a continuous time process out of it?

Rough idea: just rescale $W_n(t)$ "=" $\frac{S_{nt}}{\sqrt{n}} = \sqrt{t}\,\frac{S_{nt}}{\sqrt{nt}} \longrightarrow_w \mathcal{N}(0,t)$.

Doesn't make sense if $t \notin \mathbb{N}$ in general.
But we can fix that.

Def: Given $\{X_n\}_{n=1}^{\infty}$ as above, define for $t \geq 0$

$$W_n(t) := \frac{1}{\sqrt{n}} S_{\lfloor nt \rfloor} \qquad \lfloor a \rfloor = \sup\{n \in \mathbb{Z} : n \leq a\}$$



vertical (space) scaled $\frac{1}{\sqrt{n}}$

horizontal (time) scaled $\frac{1}{n}$

Prop: The processes $(W_n(t))_{t\geq 0}$ converge to Brownian motion $(B(t))_{t\geq 0}$ as $n\to\infty$, in finite-dimensional distributions.

Pf. We will work with increments. Fix $t > s \geq 0$.

terms $= \lfloor nt \rfloor - \lfloor ns \rfloor$. $\to \infty$ as $n\to\infty$.

$$W_n(t) - W_n(s) = \frac{1}{\sqrt{n}}\left(S_{\lfloor nt \rfloor} - S_{\lfloor ns \rfloor}\right) = \frac{1}{\sqrt{n}}\left(\sum_{k=1}^{\lfloor nt \rfloor} X_k - \sum_{k=1}^{\lfloor ns \rfloor} X_k\right) = \frac{1}{\sqrt{n}} \sum_{\lfloor ns \rfloor < k \leq \lfloor nt \rfloor} X_k .$$

By the CLT $\underbrace{\frac{1}{\sqrt{\lfloor nt \rfloor - \lfloor ns \rfloor}} \cdot \sum_{\lfloor ns \rfloor < k \leq \lfloor nt \rfloor} X_k}_{Z_n(s,t)} \xrightarrow{w} \mathcal{N}(0,1)$

$$W_n(t) - W_n(s) = \frac{\sqrt{\lfloor nt \rfloor - \lfloor ns \rfloor}}{\sqrt{n}} Z_n(s,t)$$

$$\frac{\lfloor nt \rfloor}{n} = \frac{nt - ((nt))}{n} = t - \frac{\overset{\in[0,1]}{\cdot}}{n} \to t \text{ as } n\to\infty.$$

$$\frac{\sqrt{\lfloor nt \rfloor - \lfloor ns \rfloor}}{\sqrt{n}} \xrightarrow{n\to\infty} \sqrt{t-s} \qquad \therefore \; W_n(t) - W_n(s) \xrightarrow{w} \mathcal{N}(0, t-s).$$

So, the increments of $(W_n(t))_{t\geq 0}$ are asymptotically normally distributed:

For $0 \leq s < t$, $W_n(t) - W_n(s) \xrightarrow[w]{n\to\infty} \mathcal{N}(0, t-s) \overset{d}{=} B(t) - B(s)$.

Moreover, note that $W_n(t) - W_n(s) = \frac{1}{\sqrt{n}} \sum_{\lfloor ns \rfloor < k \leq \lfloor nt \rfloor} X_k$ (independent from $\{X_1, X_2, \dots, X_{\lfloor ns \rfloor}\}$)

$\therefore$ indep from $\mathcal{F}_s^{W_n}$.

Now, let $0 = t_0 < t_1 < \dots < t_k$.

We know $(W_n(t_j) - W_n(t_{j-1}))_{j=1}^{k}$ are independent & $(B(t_j) - B(t_{j-1}))_{j=1}^{k}$ are independent.

On the last slide, we proved that

$$W_n(t_j) - W_n(t_{j-1}) \xrightarrow[w]{n\to\infty} B(t_j) - B(t_{j-1}) \qquad j = 1 \dots k$$

Thus, by the last proposition in [Lec. 53.1],

$$(W_n(t_1) - W_n(t_0), \dots, W_n(t_k) - W_n(t_{k-1}))$$
$$\xrightarrow[w]{n\to\infty} (B(t_1) - B(t_0), \dots, B(t_k) - B(t_{k-1})).$$

$$\left(W_n(t_1)-\cancel{W_n(t_0)}, \ldots, W_n(t_k)-W_n(t_{k-1})\right) \xrightarrow[w]{n\to\infty} \left(B(t_1)-\cancel{B(t_0)}, \ldots, B(t_k)-B(t_{k-1})\right).$$

Consider the function $f: \mathbb{R}^k \to \mathbb{R}^k$

$$f(x_1, x_2, \ldots, x_k) = (x_1, x_1+x_2, x_1+x_2+x_3, \ldots, x_1+x_2+\cdots+x_k)$$

By the continuous mapping theorem,

$$f\left(W_n(t_1)-W_n(t_0), \ldots, W_n(t_k)-W_n(t_{k-1})\right) \xrightarrow[w]{n\to\infty} f\left(B(t_1)-B(t_0), \ldots, B(t_k)-B(t_{k-1})\right)$$

$$\overset{\parallel}{(W_n(t_1), W_n(t_2), \ldots, W_n(t_k))} \xrightarrow[w]{} \overset{\parallel}{(B(t_1), B(t_2), \ldots, B(t_k))}$$

Finally, if the $\{t_j\}_{j=1}^{k}$ are not indexed in increasing order, permute them and permute the variables of f accordingly, at the beginning and at the end.

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So, $W_n \xrightarrow{f.d.} B$. What about $W_n \to_w B$?

Tightness? Oops: not conts. continuous.