We skipped a few standard results about weak convergence of random vectors in $\mathbb{R}^d$, which we'll need now.

Theorem: (Slutzky) Let $(X_n)_{n\in\mathbb{N}} \in \mathbb{R}^d$ and $(Y_n)_{n\in\mathbb{N}} \in \mathbb{R}^m$ be random vectors defined on the same probability space. If $X_n \to_w X$ (random) and $Y_n \to_w a$ (constant), then $(X_n, Y_n) \to_w (X, a)$.

(The law of (X_n, Y_n) is not defined here – the point is that there is only one way to couple (X, a), and so the result doesn't care about the coupling of (X_n, Y_n).)

First we need a

Lemma: For a constant $a \in \mathbb{R}^m$, $Y_n \to_w a$ iff $Y_n \to_{\mathbb{P}} a$.

Pf. $(\Leftarrow)$ [Lec 22.1]

$(\Rightarrow)$ Portmanteau theorem: $\Downarrow$ if F is closed, $\limsup_{n\to\infty} \mathbb{P}(Y_n \in F) \leq \mathbb{P}(a \in F)$

Pf. (of Slutzky's theorem) By the Portmanteau theorem, suffices to show

$$\lim_{n\to\infty} \mathbb{E}[f(X_n, Y_n)] = \mathbb{E}[f(X, a)] \quad \forall f \in$$

$$|\mathbb{E}[f(X_n, Y_n) - f(X_n, a)]|$$

$$\therefore \limsup_{n\to\infty} |\mathbb{E}[f(X_n, Y_n) - f(X_n, a)]|$$

Also, since $X_n \to_w X$, $\mathbb{E}[f(X_n, a)] \xrightarrow{n\to\infty} \mathbb{E}[f(X, a)]$.

$$\therefore \limsup_{n\to\infty} |\mathbb{E}[f(X_n, Y_n) - f(X, a)]|$$

$$\leq \limsup_{n\to\infty} |\mathbb{E}[f(X_n, Y_n) - f(X_n, a)]| + \limsup_{n\to\infty} |\mathbb{E}[f(X_n, a) - f(X, a)]|$$

Cor: (a) If $X_n, Y_n \in \mathbb{R}^d$ with $X_n \to_w X$, $Y_n \to_w a \equiv$ const, then $X_n + Y_n \to_w X + a$.

(b) If $X_n \in \mathbb{R}^d$, $Y_n \in \mathbb{R}$ with $X_n \to_w X$, $Y_n \to_w a \equiv$ const, then $X_n Y_n \to_w aX$.

Pf.

There is one coupling for which Slutzky's result holds, even if $Y_n \to_w Y \not\equiv$ const.

Prop: Let $\mu_n, \mu \in \mathrm{Prob}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ and $\nu_n, \nu \in \mathrm{Prob}(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$

If $\mu_n \to_w \mu$ and $\nu_n \to_w \nu$, then $\mu_n \otimes \nu_n \to_w \mu \otimes \nu$.

Pf. By Skorohod's theorem, construct probability spaces $(\Omega_j, \mathcal{F}_j, \mathbb{P}_j)$ $j=1,2$ and random variables

$$X_n, X : (\Omega_1, \mathcal{F}_1, \mathbb{P}_1) \to \mathbb{R}^d \quad \mathrm{Law}(X_n) = \mu_n, \ \mathrm{Law}(X) = \mu$$

$$Y_n, Y : (\Omega_2, \mathcal{F}_2, \mathbb{P}_2) \to \mathbb{R}^n \quad \mathrm{Law}(Y_n) = \nu_n, \ \mathrm{Law}(Y) = \nu$$

s.t. $X_n \to X$ $\mathbb{P}_1$-a.s. $Y_n \to Y$ $\mathbb{P}_2$-a.s.