We skipped a few standard results about weak convergence of random vectors in $\mathbb{R}^d$, which we'll need now.

**Theorem:** (Slutzky) Let $(X_n)_{n\in\mathbb{N}} \in \mathbb{R}^d$ and $(Y_n)_{n\in\mathbb{N}} \in \mathbb{R}^m$ be random vectors defined on the same probability space. If $X_n \to_w X$ (random) and $Y_n \to_w a$ (constant), then $(X_n, Y_n) \to_w (X, a)$.

(The law of $(X_n, Y_n)$ is not defined here — the point is that there is only one way to couple $(X, a)$, and so the result doesn't care about the coupling of $(X_n, Y_n)$.)

First we need a

**Lemma:** For a constant $a \in \mathbb{R}^m$, $Y_n \to_w a$ iff $Y_n \to_{\mathbb{P}} a$.

Pf. $(\Leftarrow)$ [Lec 22.1]

$(\Rightarrow)$ Portmanteau theorem: $\Downarrow$ if $F$ is closed, $\limsup_{n\to\infty} \mathbb{P}(Y_n \in F) \le \mathbb{P}(a \in F)$

$$\limsup_{n\to\infty} \mathbb{P}(\|Y_n - a\| > \varepsilon) = \limsup_{n\to\infty} \mathbb{P}(Y_n \in \underbrace{B_\varepsilon(a)^c}_{\text{closed}}) \le \mathbb{P}(a \in B_\varepsilon(a)^c) = 0. \quad ///$$

Pf. (of Slutzky's theorem) By the Portmanteau theorem, suffices to show

$$\lim_{n\to\infty} \mathbb{E}[f(X_n, Y_n)] = \mathbb{E}[f(X, a)] \quad \forall f \in \mathrm{Lip}_b(\mathbb{R}^d \times \mathbb{R}^m).$$

$$|\mathbb{E}[\underbrace{f(X_n,Y_n) - f(X_n,a)}_{D_n}]| \le \underbrace{\mathbb{E}[D_n : \|Y_n - a\| \le \varepsilon]}_{\le \|f\|_{Lip}\,\varepsilon} + \underbrace{\mathbb{E}[D_n : \|Y_n - a\| > \varepsilon]}_{\le 2\|f\|_\infty \mathbb{P}(\|Y_n - a\| > \varepsilon)} \quad \forall \varepsilon > 0.$$

$\to 0$ as $n \to \infty$.

$$\therefore \limsup_{n\to\infty} |\mathbb{E}[f(X_n,Y_n) - f(X_n,a)]| \le \cancel{\|f\|_{Lip}\,\varepsilon}^{\,0} \quad \forall \varepsilon > 0.$$

Also, since $X_n \to_w X$, $\mathbb{E}[f(X_n,a)] \xrightarrow{n\to\infty} \mathbb{E}[f(X,a)]$.

$$\therefore \limsup_{n\to\infty} |\mathbb{E}[f(X_n,Y_n) - f(X,a)]|$$

$$\le \underbrace{\limsup_{n\to\infty} |\mathbb{E}[f(X_n,Y_n) - f(X_n,a)]|}_{=0} + \underbrace{\limsup_{n\to\infty} |\mathbb{E}[f(X_n,a) - f(X,a)]|}_{=0}$$

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**Cor**: (a) If $X_n, Y_n \in \mathbb{R}^d$ with $X_n \to_w X$, $Y_n \to_w a \equiv$ Const, then $X_n + Y_n \to_w X + a$.

(b) If $X_n \in \mathbb{R}^d$, $Y_n \in \mathbb{R}$ with $X_n \to_w X$, $Y_n \to_w a \equiv$ Const, then $X_n Y_n \to_w aX$.

Pf. By Slutzky's theorem,

$$(X_n, Y_n) \to_w (X, a)$$

(a) If $f: \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^m$ is continuous, $f(X_n, Y_n) \to_w f(X, a)$ by Cont. Mapping Thm,

apply w $f(x,y) = x+y$.

(b) $g: \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^m$ continuous, $g(X_n, Y_n) \to_w g(X, a)$ " "

$g(x,y) = xy$.

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There is <u>one</u> coupling for which Slutzky's result holds, even if $Y_n \to_w Y \not\equiv$ const.

**Prop:** Let $\mu_n, \mu \in \mathrm{Prob}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ and $\nu_n, \nu \in \mathrm{Prob}(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$

If $\mu_n \to_w \mu$ and $\nu_n \to_w \nu$, then $\mu_n \otimes \nu_n \to_w \mu \otimes \nu$.

**Pf.** By Skorohod's theorem, construct probability spaces $(\Omega_j, \mathcal{F}_j, \mathbb{P}_j)$ $j=1,2$ and random variables

$$X_n, X : (\Omega_1, \mathcal{F}_1, \mathbb{P}_1) \to \mathbb{R}^d \qquad \mathrm{Law}(X_n) = \mu_n, \ \mathrm{Law}(X) = \mu$$

$$Y_n, Y : (\Omega_2, \mathcal{F}_2, \mathbb{P}_2) \to \mathbb{R}^n \qquad \mathrm{Law}(Y_n) = \nu_n, \ \mathrm{Law}(Y) = \nu$$

s.t. $X_n \to X$ $\mathbb{P}_1$-a.s. $Y_n \to Y$ $\mathbb{P}_2$-a.s.

Let $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mathbb{P}_1 \otimes \mathbb{P}_2)$ $\qquad \mathrm{Law}(Z_n) = \mu_n \otimes \nu_n$

$Z_n := (X_n, Y_n) : (\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}^d \times \mathbb{R}^n$ $\qquad \mathrm{Law}(Z) = \mu \otimes \nu$.

$Z := (X, Y)$

$\therefore Z_n \to Z$ a.s. $\mathbb{P}$

$\therefore \forall f \in C_b(\mathbb{R}^d \times \mathbb{R}^n)$ $f(Z_n) \to f(Z)$ a.s. $[\mathbb{P}]$.

$$\int f \, d(\mu_n \otimes \nu_n) = \mathbb{E}[f(Z_n)] \xrightarrow{\text{DCT}} \mathbb{E}[f(Z)] = \int f \, d(\mu \otimes \nu). \quad ///$$