Tightness originally arose in discussions around weak convergence. In the context of measures on path space, it exactly bridges the gap between weak convergence, and convergence of f.d. distributions.

Def: Let $(X_n(t))_{t\geq 0, n\leq\infty}$ be stochastic processes $(\Omega, \mathcal{F}, \mathbb{P}) \to (S, \mathcal{B})$.
Say $X_n \to_{f.d.} X_\infty$ (convergence of finite-dimensional distributions)
if $\forall k \in \mathbb{N}, \forall t_1, \dots, t_k \geq 0$ $\quad (X_n(t_1), \dots, X_n(t_k)) \to_w (X_\infty(t_1), \dots, X_\infty(t_k))$

I.e. $\forall f \in C_b(S^k, \mathbb{R})$,
$$\mathbb{E}[f(X_n(t_1), \dots, X_n(t_k))] \xrightarrow{n\to\infty} \mathbb{E}[f(X_\infty(t_1), \dots, X_\infty(t_k))].$$

Theorem: Let $(X_n(t))_{t\geq 0, n\leq\infty}$ be continuous stochastic processes in a separable, complete metric space S. Then: $\quad X_n \to_w X_\infty \quad$ I.e. $\mathbb{E}[F(X_n)] \to \mathbb{E}[F(X_\infty)]$ $\forall F \in C_b(C([0,1], S))$

iff $\quad X_n \to_{f.d.} X_\infty$

& $\{P_{X_n}\}_{n\in\mathbb{N}}$ is tight.

Pf. ($\Rightarrow$) If $X_n \to_w X_\infty$, then $\{X_n\}_{n\in\mathbb{N}}$ is tight. This is the converse of Prohorov's theorem [Lec 23.2]. We proved this in the special case $X_n, X_\infty \in \mathbb{R}^d$ in [Lec 23.1]; see [Driver, §28.6.4] for details.

If $X_n \to_w X_\infty$, then $\mathbb{E}[F(X_n)] \to \mathbb{E}[F(X_\infty)]$ $\forall F \in C_b(C([0,1],S))$.

In particular, for $t_1,\dots,t_k \geq 0$ and $f \in C_b(S^k)$,

$$F(\omega) = f(\omega(t_1),\dots,\omega(t_k)) \quad \text{"cylinder function"}.$$

defines a bounded continuous function of $\omega \in C([0,1],S)$.

$$d_k(\underline{x},\underline{y}) = \max_{1\leq j\leq k} d_S(x_j,y_j).$$

$f \in C_b(S^k)$: $\forall \epsilon>0, \exists \delta>0$ s.t. $d_k(\underline{x},\underline{y})<\delta \Rightarrow |f(\underline{x})-f(\underline{y})|<\epsilon$.

So if $\omega,\omega' \in C([0,1],S)$ w/ $d_\infty(\omega,\omega')<\delta$, $\quad |F(\omega)-F(\omega')|<\epsilon$.

$$\Uparrow$$

$$d_k((\omega(t_1),\dots,\omega(t_k)),(\omega'(t_1),\dots,\omega'(t_k)))$$
$$= \max_j d(\omega(t_j),\omega'(t_j)) \leq \sup_{t\geq 0} d(\omega(t),\omega'(t)) = d_\infty(\omega,\omega')<\delta.$$

$$\therefore \mathbb{E}[f(X_n(t_1),\dots,X_n(t_k))] = \mathbb{E}[F(X_n)] \longrightarrow \mathbb{E}[F(X_\infty)] = \mathbb{E}[f(X_\infty(t_1),\dots,X_\infty(t_k))]$$

($\Leftarrow$) S'pose $\{X_n\}_{n\in\mathbb{N}}$ is tight, and $X_n \to_{f.d.} X_\infty$.

For a contradiction, we suppose $X_n \not\to_w X_\infty$; so $\exists\ \varepsilon>0$ and $F\in C_b(C([0,1],S))$

s.t. $|\mathbb{E}[F(X_{n_k})]-\mathbb{E}[F(X_\infty)]| \geq \varepsilon$ for some subsequence $n_k \uparrow \infty$.

By the assumed tightness, by Prohorov's theorem (applies b/c S is separable, $\therefore$ so is $C([0,1],S)$)

$\exists$ further subsequence $k_\ell \uparrow \infty$ and a r.v. Y taking values in $C([0,1],S)$ s.t.

$$X_{n_{k_\ell}} \to_w Y. \Rightarrow X_{n_{k_\ell}} \to_{f.d.} Y$$

$$\therefore\ |\mathbb{E}[F(Y)]-\mathbb{E}[F(X_\infty)]| = \lim_{\ell\to\infty} |\mathbb{E}[F(X_{n_{k_\ell}})]-\mathbb{E}[F(X_\infty)]| \geq \varepsilon.$$

But $X_n \to_{f.d.} X_\infty$, so $X_{n_{k_\ell}} \to_{f.d.} X_\infty$.

Claim: If two continuous processes X,Y have the same f.d. distributions, they have the same (Borel) probability distribution. In particular, $\mathbb{E}[F(X)]=\mathbb{E}[F(Y)]$ $\forall F\in C_b(C[0,1],S))$. [HW]

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