Recall that a sequence $\{\mu_n\}_{n\in\mathbb{N}}$ of probability measures on a common measurable metric space $(C, \mathcal{B})$ is tight if, for each $\varepsilon>0$, $\exists$ compact $K_\varepsilon \subseteq C$ s.t. $\mu_n(K_\varepsilon^c) < \varepsilon$ $\forall n \in \mathbb{N}$.

Let $(X_n(t))_{t\in[0,1]}$ be a sequence of continuous stochastic processes in S.
Their laws are $P_n \in \mathrm{Prob}(C([0,1],S), \mathcal{B}(C([0,1],S)))$
When is such a sequence tight?

First question: what do compact sets in $C([0,1],S)$ look like?

Theorem: (Arzela-Ascoli) Let S be a complete metric space w Heine-Borel property (closed bounded sets are compact). Then a subset $K \subseteq C([0,1],S)$, d_∞
is compact iff it is closed, pointwise bounded, and equicontinuous:

$\exists \beta: [0,1] \to \mathbb{R}_+$ s.t. $d(x_0, \omega(s)) \leq \beta(s)$ $\forall s \in [0,1]$.

$\forall t_0 \in [0,1]$, $\omega \in K$, $\varepsilon > 0$ $\exists \delta = \delta(t_0, \varepsilon) > 0$ s.t.

$$\forall s \in [0,1], \quad |s-t_0| < \delta \implies d_S(\omega(s), \omega(t_0)) < \varepsilon.$$

Lemma: Let $N < \infty$, $\alpha > 0$. S'pose $W \subseteq C([0,1], S)$ satisfies

$$d_S(w(s), w(t)) \leq N|s-t|^\alpha \quad \forall s,t \in [0,1],$$

and $$d_S(w(0), x_0) \leq N \quad \text{for some } x_0 \in S.$$

Then W is uniformly equicontinuous and uniformly bounded.

Pf. Equicontinuous: $\delta = (\varepsilon/N)^{1/\alpha}$

$\forall s,t \quad |s-t| < \delta \Rightarrow d_S(w(s), w(t)) \leq N|s-t|^\alpha < N\delta^\alpha = \varepsilon.$

Bounded: $\forall w \in W,\ t \in [0,1]$

$$d(x_0, w(t)) \leq d(x_0, w(0)) + d(w(0), w(t))$$
$$\leq N + N\underbrace{|t-0|^\alpha}_{\leq N} \leq 2N. \quad ///$$

Note: the d_∞-closure of an equicontinuous / pointwise bounded set is equicontinuous / pointwise bounded.

Theorem: (Kolmogorov's Tightness Criteria)

Let S be a complete metric space with the Heine-Borel property. Let $(X_n(t))_{t\in[0,1], n\in\mathbb{N}}$ be a sequence of continuous stochastic processes in S. S'pose $\exists\, \varepsilon, C > 0$ and $p \geq 1+\varepsilon$ s.t.

$$\sup_n \mathbb{E}[d_S(X_n(s), X_n(t))^p] \leq C|s-t|^{1+\varepsilon} \quad \forall s,t\in[0,1],$$

and $\{X_n(0)\}_{n\in\mathbb{N}}$ is uniformly bounded w high probability

i.e. is tight.

$$\lim_{N\to\infty} \sup_n \mathbb{P}(\underbrace{d_S(X_n(0), x_0) > N}_{X_n(0) \in \overline{B_N(x_0)}^c}) = 0 \text{ for some } x_0 \in S.$$

Then the laws $\{P_n\}_{n\in\mathbb{N}} \subset \mathrm{Prob}(C([0,1],S))$ form a tight sequence of probability measures on path space.

Pf. Recall $\Delta_k(\omega) = \max_{1\le j\le 2^k} d\left(\omega\left(\frac{j-1}{2^k}\right), \omega\left(\frac{j}{2^k}\right)\right)$

$$K_\alpha(\omega) = 2^{1+\alpha} \sum_{k=0}^{\infty} 2^{\alpha k} \Delta_k(\omega) \qquad K_\alpha(X_n) = K_\alpha(t \mapsto X_n(t))$$

By Kolmogorov's Continuity Criteria, for any $\alpha \in (0, \varepsilon/p)$, $K_\alpha(X_n) \in L^p$:

$$\mathbb{E}[K_\alpha(X_n)^p] \le \frac{C \cdot 2^{p(1+\alpha)}}{(1-2^{\alpha-\varepsilon/p})^p} =: M < \infty .$$

Fix any $\alpha \in (0, \varepsilon/p)$, and define

$$W_N^\alpha = \{\omega \in C([0,1], S) : K_\alpha(\omega) \le N \ \& \ d(\omega(0), x_0) \le N\}$$

Then $P_n((W_N^\alpha)^c) = \mathbb{P}(X_n \notin W_N^\alpha)$

$$= \mathbb{P}(K_\alpha(X_n) > N \text{ or } d(X_n(0), x_0) > N)$$

$$\le \mathbb{P}(K_\alpha(X_n) > N) + \underbrace{\mathbb{P}(d(X_n(0), x_0) > N)}$$

Markov's Ineq:

$$\le \frac{1}{N^p} \mathbb{E}[K_\alpha(X_n)^p] \le \frac{M}{N^p} \to 0 \text{ as } N \to \infty . \qquad \le \sup_m \mathbb{P}(d(X_m(0), x_0) > N) \to 0 \text{ as } N \to \infty .$$

Thus $\sup_n P_n((W_N^\alpha)^c) \to 0$ as $N \to \infty$.

Now set $K_N^\alpha = \overline{W_N^\alpha}$. Then $W_N^\alpha \subseteq K_N^\alpha \therefore (K_N^\alpha)^c \subseteq (W_N^\alpha)^c$

$$\sup_n P_n((K_N^\alpha)^c) \leq \sup_n P_n((W_N^\alpha)^c) \to 0 \text{ as } N \to \infty.$$

Since W_N^α is uniformly equicontinuous and bounded (by the lemma), it follows that $\{K_N^\alpha\}_{N \in \mathbb{N}}$ are all compact. $\therefore \{P_n\}_{n \in \mathbb{N}}$ is tight. ///