Probability Measures on Path Space

Let (Xt) teco,11 be a Continuous - path process in a metric spale S

 $(X_t)_{t\in [0,1]}: (\mathcal{D}_{1}, \mathcal{F}_{1}, \mathbb{P}) \to (S, \mathcal{B}(S))$

 $(t \mapsto X_{t}(w))_{t \in [0,1]} \in C([0,1],S)$

Thanks to Kolmogorov, we know such things exist.

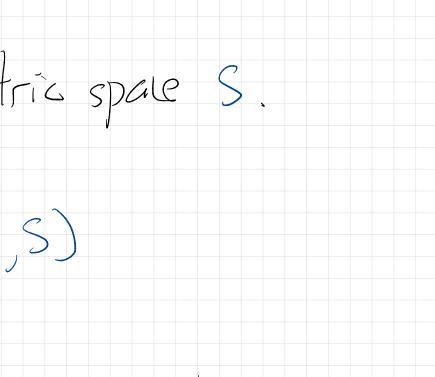
Start with a process (Yt)tergin satisfying the Kolmogorov Criteria; select a version (Yt)tergin that is a.s. continuous

The law of such a process is therefore a probability measure on path space

 $P_X(E) = P_{\Sigma} = (b \rightarrow X_{U})_{b \in [0,1]} \in E_{\Sigma}$

 $\rightarrow E \subseteq C(E_0,11,S)$

What 5-field should we take?



Def: The Cylinder 5-Field C=C(L911, S) is the 5-field generated by the projections $\pi_t : C([n],S) \rightarrow S : \pi_t(w) = \omega(t)$.

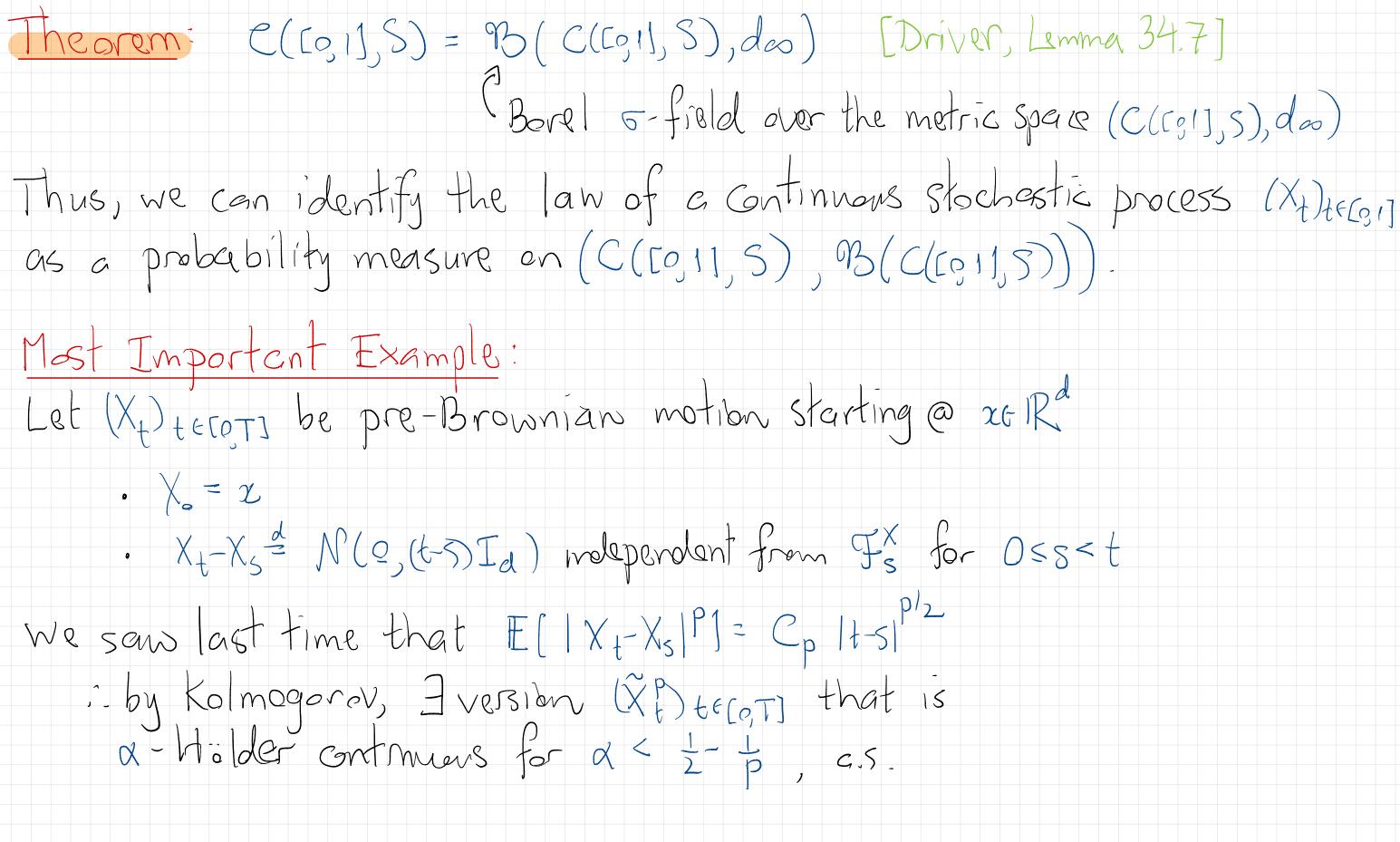
Eg. If neN, BEM(Sn), and time [0,1],

The path space is a metric space in its own right.

 $d_{\infty}: C(E_0,11,5)^2 \to E_0,\infty)$

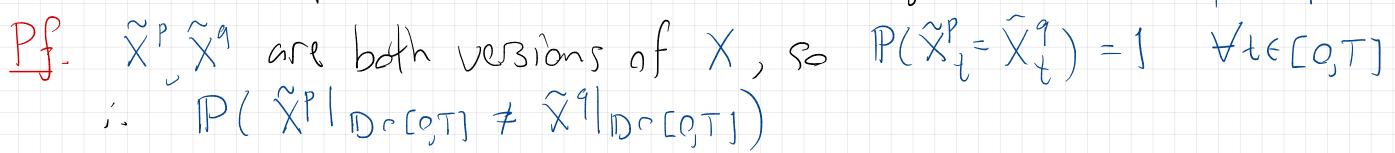
 $d_{\infty}(w, \eta) :=$

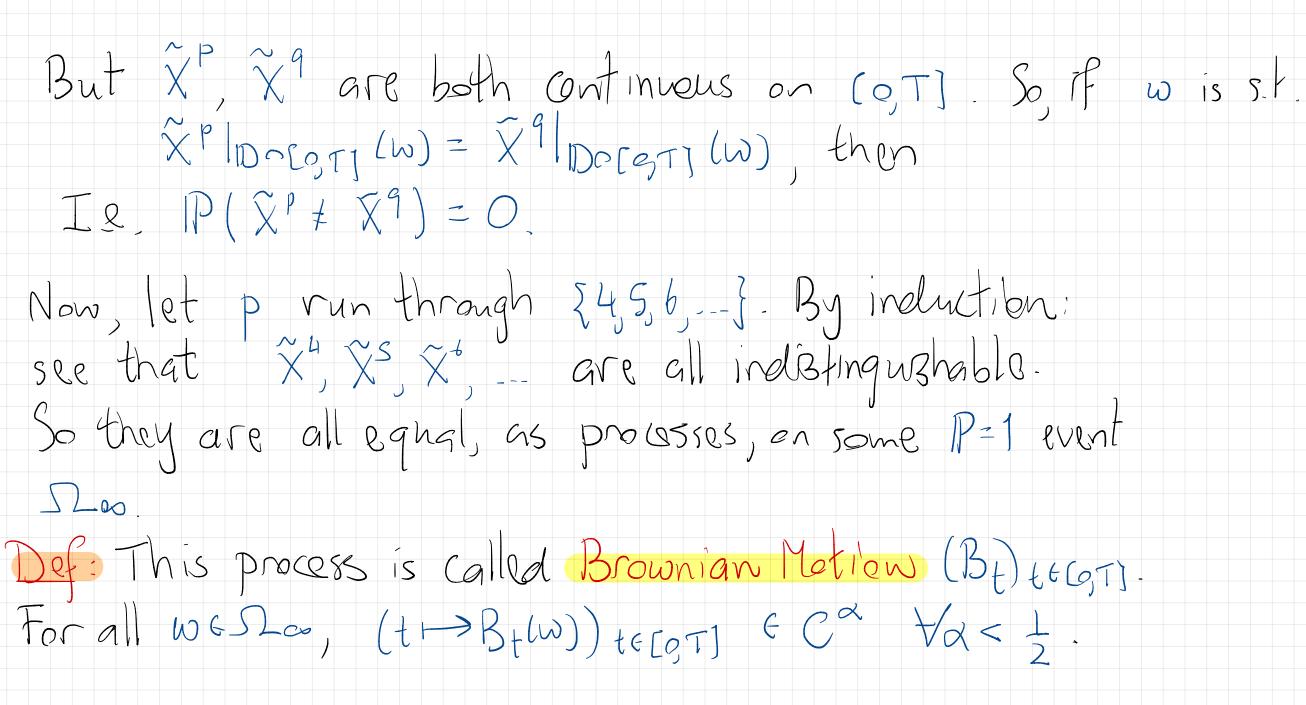
This is a complete metric (even if ds is not), and is separable iff S is separable.



Since $P(S_p) = 1$, $P(S_p \cap S_q) = 1$, $\forall p, q > 2$

Lemma. The processes XP, Xª are indistinguishable on sporg.







Def. The Wiener measure W_T^{χ} is the law of Brownian Metion:

 $W_T \in Prob(C(IO_T], \mathbb{R}^d), C(IO_T], \mathbb{R}^d))$

 $W_T^{\infty}(E) = P((t \mapsto B_t)_{t \in [0,T]} \in E \mid B_e = x)$

This measure was originally constructed by N. Wiener @ MIT > in 1923 (age 29), almost 15 years before Kolmogorov and his school set probability theory on rigorous footing, using ideas really engineered by Wiener.

l None of the tools we've used this year existed.

Wiener more directly constructed this measure on

path spale, using the Daniel integral (introduced

4 years earlier). From a modern viewpoint, Wiener

defined the process through its (random) Fourier series, which he masterfully slowed is $C^{\alpha}(\alpha < \frac{1}{2})$ with

delicate Convergence orguments.