Probability Measures on Path Space

Let $(X_t)_{t\in[0,1]}$ be a continuous-path process in a metric space S.

$$(X_t)_{t\in[0,1]} : (\Omega, \mathcal{F}, \mathbb{P}) \to (S, \mathcal{B}(S))$$

$$(t \mapsto X_t(\omega))_{t\in[0,1]} \in C([0,1],S)$$

Thanks to Kolmogorov, we know such things exist.

↳ Start with a process $(Y_t)_{t\in[0,1]}$ satisfying the Kolmogorov Criteria; select a version $(\tilde{Y}_t)_{t\in[0,1]}$ that is a.s. continuous on $\Omega_0 \subset \Omega$

$X_t = \tilde{Y}_t|_{\Omega_0}$ $\quad \mathbb{P}(\Omega_0) = 1.$

The law of such a process is therefore a probability measure on path space

$$\mathbb{P}_X(E) = \mathbb{P}\{\omega\in\Omega : (t\mapsto X_t(\omega))_{t\in[0,1]} \in E\}$$

$$E \subseteq C([0,1],S)$$

What σ-field should we take?

Def: The Cylinder σ-Field $\mathcal{C} = \mathcal{C}([0,1],S)$ is the σ-field generated by the projections $\pi_t : C([0,1],S) \to S : \pi_t(\omega) = \omega(t)$.

$$\neq S^{[0,1]}$$

$$\mathcal{C} = \sigma\big(\pi_t^{-1}(B) : B \in \mathcal{B}(S)\big)$$

$$\pi_t^{-1}(B) = \{\omega \in C([0,1],S) : \omega(t) \in B\}$$

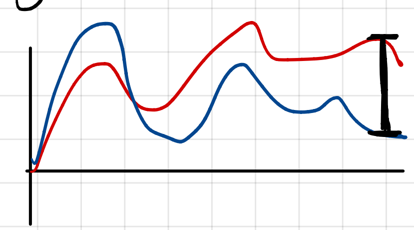
Eg. If $n \in \mathbb{N}$, $B \in \mathcal{B}(S^n)$, and $t_1, \dots, t_n \in [0,1]$,

$$\{\omega \in C([0,1],S) : (\omega(t_1), \dots, \omega(t_n)) \in B\} \in \mathcal{C}([0,1],S).$$

The path space is a metric space in its own right.

$$d_\infty : C([0,1],S)^2 \to [0,\infty)$$

$$d_\infty(\omega,\eta) := \overset{\max}{\cancel{\sup}}_{0 \le t \le 1} d_S(\omega(t), \eta(t))$$



This is a complete metric (even if d_S is not), and is separable iff S is separable.

Theorem: $\mathcal{C}([0,1],S) = \mathcal{B}(C([0,1],S), d_\infty)$ [Driver, Lemma 34.7]

Borel σ-field over the metric space $(C([0,1],S), d_\infty)$

Thus, we can identify the law of a continuous stochastic process $(X_t)_{t\in[0,1]}$ as a probability measure on $(C([0,1],S), \mathcal{B}(C([0,1],S)))$.

Most Important Example:

Let $(X_t)_{t\in[0,T]}$ be pre-Brownian motion starting @ $x \in \mathbb{R}^d$

- $X_0 = x$
- $X_t - X_s \overset{d}{=} \mathcal{N}(0, (t-s)I_d)$ independent from $\mathcal{F}_s^X$ for $0 \le s < t$

We saw last time that $\mathbb{E}[|X_t - X_s|^p] = C_p |t-s|^{p/2}$

$\therefore$ by Kolmogorov, $\exists$ version $(\tilde{X}_t^p)_{t\in[0,T]}$ that is α-Hölder continuous for $\alpha < \frac{1}{2} - \frac{1}{p}$, a.s.

$\hookrightarrow$ I.e. $\exists\, \Omega_p \subseteq \Omega$, $\mathbb{P}(\Omega_p) = 1$ s.t. $t \mapsto \tilde{X}_t^p(\omega)$ is C^α, $\forall \omega \in \Omega_p$.

Since $\mathbb{P}(\Omega_p)=1$, $\mathbb{P}(\Omega_p \cap \Omega_q)=1$ $\forall p,q>2$

Lemma: The processes $\tilde{X}^p, \tilde{X}^q$ are indistinguishable on $\Omega_p \cap \Omega_q$.

Pf. $\tilde{X}^p, \tilde{X}^q$ are both versions of X, so $\mathbb{P}(\tilde{X}^p_t = \tilde{X}^q_t)=1$ $\forall t\in[0,T]$.

$$\therefore \quad \mathbb{P}\left(\tilde{X}^p|_{\mathbb{D}\cap[0,T]} \neq \tilde{X}^q|_{\mathbb{D}\cap[0,T]}\right)$$

$$= \mathbb{P}\Big(\bigcup_{t\in\mathbb{D}\cap[0,T]} \{\tilde{X}^p_t \neq \tilde{X}^q_t\}\Big) \leq \sum_{t\in\mathbb{D}\cap[0,T]} \underbrace{\mathbb{P}(\tilde{X}^p_t \neq \tilde{X}^q_t)}_{\downarrow} = 0.$$

But $\tilde{X}^p, \tilde{X}^q$ are both continuous on $[0,T]$. So, if ω is s.t.

$\tilde{X}^p|_{\mathbb{D}\cap[0,T]}(\omega) = \tilde{X}^q|_{\mathbb{D}\cap[0,T]}(\omega)$, then $\tilde{X}^p_t(\omega) = \tilde{X}^q_t(\omega)$ $\forall t\in[0,T]$.

I.e. $\mathbb{P}(\tilde{X}^p \neq \tilde{X}^q)=0$. $\quad C^\alpha$ $\alpha<\frac{1}{2}-\frac{1}{p}$ $\quad \alpha<\frac{1}{2}-\frac{1}{q}$,

Now, let p run through $\{4,5,6,\dots\}$. By induction: see that $\tilde{X}^4, \tilde{X}^5, \tilde{X}^6, \dots$ are all indistinguishable.
So they are all equal, as processes, on some $\mathbb{P}=1$ event Ω_∞.

Def: This process is called **Brownian Motion** $(B_t)_{t\in[0,T]}$.
For all $\omega\in\Omega_\infty$, $(t\mapsto B_t(\omega))_{t\in[0,T]} \in C^\alpha$ $\forall \alpha<\frac{1}{2}$.

Def: The Wiener measure W_T^x is the law of Brownian Motion:

$$W_T^x \in \mathrm{Prob}\big(C([0,T],\mathbb{R}^d),\ \mathcal{C}([0,T],\mathbb{R}^d)\big)$$

$$W_T^x(E) = \mathbb{P}\big((t \mapsto B_t)_{t\in[0,T]} \in E \mid B_0 = x \big)$$

This measure was originally constructed by N. Wiener @ MIT in 1923 (age 29), almost 15 years before Kolmogorov and his school set probability theory on rigorous footing, using ideas really engineered by Wiener.

None of the tools we've used this year existed.

Wiener more directly constructed this measure on path space, using the Daniel integral (introduced 4 years earlier). From a modern viewpoint, Wiener defined the process through its (random) Fourier series, which he masterfully showed is C^α $(\alpha < \frac{1}{2})$ with delicate convergence arguments.