We now have a computational way to guarantee a path $\omega : \mathbb{D} \cap [0,1] \to (S,d)$ is α-Hölder continuous:

$$K_\alpha(\omega) := 2^{1+\alpha} \sum_{n\geq 0}^{\infty} 2^{\alpha n} \Delta_n(\omega)$$

If $K_\alpha(\omega) < \infty$, then $\omega \in C^\alpha$ and $|\omega(s)-\omega(t)| \leq K_\alpha(\omega)\,|s-t|^\alpha \;\; \forall s,t \in \mathbb{D} \cap [0,1]$.

Moreover: if (S,d) is complete, then $\exists!$ C^α path $\bar{\omega} : [0,1] \to (S,d)$ (with the same constant $K_\alpha(\omega) = K_\alpha(\bar{\omega})$) extending it $\bar{\omega}|_{\mathbb{D}} = \omega$.

This gives us a "path" to guarantee the existence of a C^α version of a stochastic process, based on its 2-dimensional distributions.

Theorem: Let $(X_t)_{t\in \mathbb{D}\cap[0,1]} : (\Omega, \mathcal{F}, \mathbb{P}) \to (S,d)$

Suppose $\exists\, \varepsilon > 0$, $p > 1+\varepsilon$, and $C > 0$ s.t.

$$\mathbb{E}[d(X_s, X_t)^p] \leq C\,|s-t|^{1+\varepsilon} \quad \forall s,t \in \mathbb{D} \cap [0,1]$$

Then for each $\alpha \in (0, \varepsilon/p)$, $\exists$ r.v. $K_\alpha(X) \in L^p$ s.t.

$$d(X_s, X_t) \leq K_\alpha(X)\,|s-t|^\alpha \quad \forall s,t \in \mathbb{D} \cap [0,1].$$

Theorem: Let $(X_t)_{t\in\mathbb{D}\cap[0,1]} : (\Omega, \mathcal{F}, \mathbb{P}) \to (S, d)$ Suppose $\exists\, \varepsilon > 0,\ p \geq 1$ and $C > 0$ s.t.

$$\mathbb{E}[d(X_s, X_t)^p] \leq C|s-t|^{1+\varepsilon} \quad \forall s,t \in \mathbb{D}\cap[0,1]$$

Then for each $\alpha \in (0, \varepsilon/p)$, $\exists$ r.v. $K_\alpha(X) \in L^p$ s.t.

$$d(X_s, X_t) \leq K_\alpha(X)|s-t|^\alpha \quad \forall s,t \in \mathbb{D}\cap[0,1].$$

Pf. Set $K_\alpha(X)(\omega) = K_\alpha(t \mapsto X_t(\omega))$

$$\Delta_n(X)^p = \max_{1\leq i\leq 2^n} d\left(X_{\frac{i}{2^n}}, X_{\frac{i-1}{2^n}}\right)^p$$

$\mathbb{E}(\qquad)$

$$\therefore \|\Delta_n(X)\|_{L^p} \leq C^{1/p}\, 2^{-n\frac{\varepsilon}{p}}$$

$$\Rightarrow \|K_\alpha(X)\|_{L^p} \leq 2^{1+\alpha}\sum_{n=0}^{\infty} 2^{n\alpha}\|\Delta_n(X)\|_{L^p}$$

$$\leq 2^{1+\alpha}\sum_{n=0}^{\infty} 2^{n\alpha}\cdot 2^{-n\frac{\varepsilon}{p}} =$$

In particular, $K_\alpha(X) < \infty$ a.s.

$\Rightarrow$ on this $\mathbb{P}=1$ event, $t \mapsto X_t$ is C^α on $\mathbb{D}\cap[0,1]$,

with constant $\leq K_\alpha(X)$.

Now we need to go beyond $\mathbb{D}$, and beyond $[0,1]$

Theorem: (Kolmogorov's Continuity Criteria)

Let $T \in \mathbb{N}$. Let (S,d) be a complete, separable metric space (e.g. $\mathbb{R}^d$).
Let $(X_t)_{t\in[0,T]} : (\Omega, \mathcal{F}, \mathbb{P}) \to (S,d)$ be a stochastic process, s.t.

$$\mathbb{E}[d(X_s, X_t)^p] \leq C|s-t|^{1+\varepsilon} \quad \forall s,t \in [0,T]$$

for some $\varepsilon, C > 0$ and $p \geq 1+\varepsilon$. Then for each $\alpha \in (0, \varepsilon/p)$, there is a random variable $K_\alpha(X) \in L^p$, and a version $(\tilde{X}_t)_{t\in[0,T]}$ of $(X_t)_{t\in T}$ that is $C^\alpha([0,T], S)$, satisfying

$$d(\tilde{X}_s, \tilde{X}_t) \leq K_\alpha(X)|s-t|^\alpha \quad \forall s,t \in [0,T].$$

Pf. First, it suffices to prove the theorem separately on each interval $[\frac{n}{2}, \frac{n}{2}+1]$ for $0 \leq n < 2T$.

All the inequalities are translation invariant. So, suffices to prove the result on $[0,1]$.

By the preceding theorem, we have the r.v. $K_\alpha(X) \in L^p$ s.t.

$$d(X_s, X_t) \leq K_\alpha(X)\,|s-t|^\alpha \quad \forall s,t \in \mathbb{D} \cap [0,1] \quad . \quad (\star)$$

On the $\mathbb{P}=1$ event $\{K_\alpha(X) < \infty\}$, $\exists!$ C^α extension $(\tilde{X}_t)_{t\in[0,1]}$ of $(X_t)_{t\in\mathbb{D}\cap[0,1]}$ satisfying $(\star)$ on all of $[0,1]$. We now show $\tilde{X}$ is a version of X.

If $t \in [0,1]$, letting s vary through $[0,1]\cap\mathbb{D}$,

$$d(X_t, \tilde{X}_t)^p$$

Eg. Brownian Motion. $(B_t)_{t\geq 0} : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$

Independent increments, $B_t - B_s \overset{d}{=} \mathcal{N}^d(\underline{0}, (t-s) I_d)$ for $t > s \geq 0$

Eg. Poisson process. $(N_t)_{t\geq 0} : (\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{N}$, intensity λ.

Independent increments, $N_t - N_s \overset{d}{=} \mathrm{Poiss}(\lambda(t-s))$, for $t > s \geq 0$.

$$\therefore \mathbb{E}[|N_t - N_s|^p] = \sum_{n \geq 0} n^p \frac{(\lambda(t-s))^n}{n!} e^{-\lambda(t-s)}$$