We now have a computational way to guarantee a path $\omega : \mathbb{D} \cap [0,1] \to (S,d)$ is α-Hölder continuous:

$$K_\alpha(\omega) := 2^{1+\alpha} \sum_{n=0}^{\infty} 2^{\alpha n} \Delta_n(\omega) \leftarrow \Delta_n(\omega) = \max\{ d(\omega(s),\omega(t)) : |s-t| \le \tfrac{1}{2^n} \}$$

$s,t \in \mathbb{D}_n \cap [0,1]$

If $K_\alpha(\omega) < \infty$, then $\omega \in C^\alpha$ and $|\omega(s)-\omega(t)| \le K_\alpha(\omega)\, |s-t|^\alpha \quad \forall s,t \in \mathbb{D} \cap [0,1]$.

Moreover: if (S,d) is complete, then $\exists!$ C^α path $\bar\omega : [0,1] \to (S,d)$ (with the same constant $K_\alpha(\omega) = K_\alpha(\bar\omega)$) extending it $\bar\omega|_{\mathbb{D}} = \omega$.

This gives us a "path" to guarantee the existence of a C^α version of a stochastic process, based on its 2-dimensional distributions.

Theorem: Let $(X_t)_{t \in \mathbb{D} \cap [0,1]} : (\Omega, \mathcal{F}, \mathbb{P}) \to (S,d)$

Suppose $\exists\, \varepsilon > 0$, $p > 1+\varepsilon$, and $C > 0$ s.t.

$$\mathbb{E}[d(X_s,X_t)^p] \le C|s-t|^{1+\varepsilon} \quad \forall s,t \in \mathbb{D} \cap [0,1]$$

Then for each $\alpha \in (0, \varepsilon/p)$, $\exists$ r.v. $K_\alpha(X) \in L^p$ s.t.

$$d(X_s,X_t) \le K_\alpha(X)\, |s-t|^\alpha \quad \forall s,t \in \mathbb{D} \cap [0,1].$$

Theorem: Let $(X_t)_{t\in\mathbb{D}\cap[0,1]}:(\Omega,\mathcal{F},\mathbb{P})\to(S,d)$ Suppose $\exists\,\varepsilon>0,\ p\geq 1+\varepsilon$ and $C>0$ s.t.

$$\mathbb{E}[d(X_s,X_t)^p]\leq C|s-t|^{1+\varepsilon}\quad\forall s,t\in\mathbb{D}\cap[0,1]$$

[HW] if $p<1+\varepsilon \Rightarrow X_t=X_0$ a.s.

Then for each $\alpha\in(0,\varepsilon/p)$, $\exists$ r.v. $K_\alpha(X)\in L^p$ s.t.

$$d(X_s,X_t)\leq K_\alpha(X)|s-t|^\alpha\quad\forall s,t\in\mathbb{D}\cap[0,1].$$

Pf. Set $K_\alpha(X)(\omega)=K_\alpha(t\mapsto X_t(\omega))=2^{1+\alpha}\sum_{n=0}^{\infty}2^{n\alpha}\underbrace{\Delta_n(t\mapsto X_t(\omega))}_{:=\Delta_n(X)(\omega)}$

$$\mathbb{E}\Big(\Delta_n(X)^p\Big)=\mathbb{E}\Big(\max_{1\leq i\leq 2^n}d(X_{\frac{i}{2^n}},X_{\frac{i-1}{2^n}})^p\Big)\leq\sum_{i=1}^{2^n}\mathbb{E}\Big(d(X_{\frac{i}{2^n}},X_{\frac{i-1}{2^n}})^p\Big)\leq\sum_{i=1}^{2^n}C\cdot\left(\frac{1}{2^n}\right)^{1+\varepsilon}=C2^{-n\varepsilon}.$$

$\therefore\ \|\Delta_n(X)\|_{L^p}\leq C^{1/p}2^{-n\frac{\varepsilon}{p}}$

$$\Rightarrow\|K_\alpha(X)\|_{L^p}\leq 2^{1+\alpha}\sum_{n=0}^{\infty}2^{n\alpha}\|\Delta_n(X)\|_{L^p}$$

$$\leq 2^{1+\alpha}\sum_{n=0}^{\infty}2^{n\alpha}\cdot 2^{-n\frac{\varepsilon}{p}}=\frac{C^{1/p}2^{1+\alpha}}{1-2^{\alpha-\varepsilon/p}}<\infty.$$

In particular, $K_\alpha(X)<\infty$ a.s.

$\Rightarrow$ on this $\mathbb{P}=1$ event, $t\mapsto X_t$ is C^α on $\mathbb{D}\cap[0,1]$, with constant $\leq K_\alpha(X)$. ///

Now we need to go beyond $\mathbb{D}$, and beyond $[0,1] \to [0,T] \to [0,\infty)$

$\mathbb{N}$? [HW]

Theorem: (Kolmogorov's Continuity Criteria)

Let $T \in \mathbb{N}$. Let (S,d) be a complete, separable metric space (e.g. $\mathbb{R}^d$).

Let $(X_t)_{t \in [0,T]} : (\Omega, \mathcal{F}, \mathbb{P}) \to (S,d)$ be a stochastic process, s.t.

$$\mathbb{E}[d(X_s, X_t)^p] \leq C|s-t|^{1+\varepsilon} \quad \forall s,t \in [0,T]$$

for some $\varepsilon, C > 0$ and $p \geq 1+\varepsilon$. Then for each $\alpha \in (0, \varepsilon/p)$, there is a random variable $K_\alpha(X) \in L^p$, and a version $(\tilde{X}_t)_{t \in [0,T]}$ of $(X_t)_{t \in T}$ that is $C^\alpha([0,T], S)$, satisfying

$$d(\tilde{X}_s, \tilde{X}_t) \leq K_\alpha(X) |s-t|^\alpha \quad \forall s,t \in [0,T].$$

Pf. First, it suffices to prove the theorem separately on each interval $[\frac{n}{2}, \frac{n}{2}+1]$ for $0 \leq n < 2T$.

All the inequalities are translation invariant. So, suffices to prove the result on $[0,1]$.

By the preceding theorem, we have the r.v. $K_\alpha(X) \in L^p$ s.t.

$$d(X_s, X_t) \leq K_\alpha(X)\,|s-t|^\alpha \quad \forall s,t \in \mathbb{D}\cap[0,1] \,. \quad (\star)$$

On the $\mathbb{P}=1$ event $\{K_\alpha(X)<\infty\}$, $\exists!$ C^α extension $(\tilde{X}_t)_{t\in[0,1]}$ of $(X_t)_{t\in\mathbb{D}\cap[0,1]}$ satisfying $(\star)$ on all of $[0,1]$. We now show $\tilde{X}$ is a version of X.

If $t\in[0,1]$, letting s vary through $[0,1]\cap\mathbb{D}$,

$$d(X_t,\tilde{X}_t)^p \leq \liminf_{s\to t}\,[\,d(X_t,X_s) + \underbrace{d(X_s,\tilde{X}_t)}_{\to 0}\,]^p$$

$$= \liminf_{\mathbb{D}\ni s\to t} d(X_t,X_s)^p$$

$$\mathbb{E}[d(X_t,\tilde{X}_t)^p] \leq \liminf_{\mathbb{D}\ni s\to t} \mathbb{E}[d(X_t,X_s)^p]$$

$$\leq \liminf_{\mathbb{D}\ni s\to t} C|s-t|^{1+\varepsilon} = 0.$$

$$\Rightarrow \mathbb{P}(X_t \neq \tilde{X}_t) = \mathbb{P}\big(d(X_t,\tilde{X}_t)^p > 0\big) = 0. \qquad ///$$

Eg. Brownian Motion. $(B_t)_{t\geq 0} : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$

Independent increments, $B_t - B_s \overset{d}{=} \mathcal{N}^d(\underline{0}, (t-s) I_d)$ for $t > s \geq 0$

$\overset{d}{=} \sqrt{t-s}\, Z$ for $Z \overset{d}{=} \mathcal{N}(0, I)$.

$\therefore \mathbb{E}[\|B_t - B_s\|^p] = |t-s|^{p/2} \underbrace{\mathbb{E}[\|Z\|^p]}_{C}$

Take $p > 2$,

$\frac{p}{2} = 1 + \underbrace{(\frac{p}{2} - 1)}_{\varepsilon} > 0$

$\therefore$ by Kolmogorov, $\exists\, C^\alpha$ version $(\tilde{B}_t)_{t\geq 0}$ for $\alpha < \varepsilon/p = \frac{1}{2} - \frac{1}{p}$.

Eg. Poisson process. $(N_t)_{t\geq 0} : (\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{N}$, intensity λ.

Independent increments, $N_t - N_s \overset{d}{=} \mathrm{Poiss}(\lambda(t-s))$, for $t > s \geq 0$.

$\therefore \mathbb{E}[|N_t - N_s|^p] = \sum_{n\geq 0} n^p \frac{(\lambda(t-s))^n}{n!} e^{-\lambda(t-s)}$

for $p \in \mathbb{N}$,

$\sim \lambda(t-s)$ for small $|t-s|$.

$\not\leq C|t-s|^{1+\varepsilon}$