Def: Let $w : [0,1] \to (S,d)$. Fix some $\alpha \in (0,1)$.
Say w is **Hölder-α continuous**, $w \in C^\alpha([0,1], S)$ if $\exists K = K(w) < \infty$
s.t. $\forall s,t \in [0,1]$, $d(w(s), w(t)) \leq K(w)|s-t|^\alpha$.

Working with uncountable families of measurable sets is challenging.
Fortunately, Hölder continuity interacts well with (countable) dense subsets.

Lemma: Let $D \subset [0,1]$ be a dense subset. Let (S,d) be a complete metric space.
Let $\alpha \in (0,1)$. If $w : D \to S$ is α-Höler continuous, then $\exists!\ \bar{w} : [0,1] \to S$
that is α-Hölder continuous (with $K(\bar{w}) = K(w)$), and s.t. $\bar{w}|_D = w$.

Pf. Fix $t \in [0,1]$; we know $\exists\ (t_n)_{t \in \mathbb{N}}$ in D s.t. $t_n \to t$.

Define $\bar{w}(t) :=$

- Exists:

- well-defined:

- $\bar{w}|_D = w$:

- Hölder continuous: fix any sequences $t_n \to t$, $s_n \to s$.

$$d(\bar{\omega}(s), \bar{\omega}(t))$$

We're going to construct Hölder continuous versions by working on a countable dense subset of $[0,1]$.

Def: The dyadic rational numbers $\mathbb{D}$ are those real numbers whose binary expansion is finitely-terminating.

$$[0,\infty) \ni t = b_0(t) + \sum_{k=1}^{\infty} \frac{b_k(t)}{2^k} \quad \text{for unique } b_0(t) \in \mathbb{N}, \; b_k(t) \in \{0,1\}, \; k \geq 1$$

$$t \in \mathbb{D} \text{ iff } b_k(t) = 0 \;\; \forall \text{ suff. large } k.$$

Notice: $\mathbb{D} = \bigcup_{n\in\mathbb{N}} \mathbb{D}_n$, where $\mathbb{D}_n = \left\{ \frac{i}{2^n},\ i\in\mathbb{Z} \right\}$.

From this (interlacing) union, we get the following useful refinement of the characterization of $\mathbb{D}$:

For any $t\in[0,\infty)$, and any n, there are unique $a_0 = a_0^{(n)} \in \mathbb{N}$, $a_k = a_k^{(n)} \in\{0,1\}$ $(k\geq 1)$ s.t.

$$t = \frac{a_0}{2^n} + \sum_{k=1}^{\infty} \frac{a_k}{2^{n+k}}$$

and $t\in\mathbb{D}_+ = \mathbb{D}\cap[0,\infty)$ iff $a_k = 0$ for all large k.

In particular, a_0 is defined by

Notice: if $\omega \in C^\gamma$ and $|s-t| \leq \varepsilon$, then $d(\omega(s), \omega(t)) \leq |s-t|^\gamma$

In particular, if $\varepsilon = 2^{-n}$, then

Def: Let $\omega : \mathbb{D} \cap [0,1] \to (S, d)$. For each n, define

$$\Delta_n(\omega) := \max\{ d(\omega(s), \omega(t)) : s, t \in \mathbb{D}_n \cap [0,1], |s-t| \leq 2^{-n} \}$$

By the note above, if $\omega \in C^\gamma$, then $\Delta_n(\omega) \leq K(\omega) 2^{-\gamma n}$.

$\therefore$ for any $\alpha \in (0, \gamma)$,

$$\sum_{n=0}^{\infty} 2^{n\alpha} \Delta_n(\omega)$$

Prop: For each $\alpha \in (0,1)$, set $K_\alpha(\omega) = 2^{1+\alpha} \sum_{n=0}^{\infty} 2^{\alpha n} \Delta_n(\omega)$.

If $\omega : \mathbb{D} \cap [0,1] \to (S, d)$ satisfies $K_\alpha(\omega) < \infty$, then $\omega \in C^\alpha$, and

$$d(\omega(s), \omega(t)) \leq K_\alpha(\omega) |s-t|^\alpha \quad \forall s, t \in \mathbb{D} \cap [0,1].$$

Pf. The core of the proof is the following

Claim. If $s, t \in \mathbb{D} \cap [0,1]$ with $2^{-(n+1)} < t-s \leq 2^{-n}$, then

$$d(w(s), w(t)) \leq 2 \sum_{k=n}^{\infty} \Delta_k(w).$$

This will complete the proof, because

$$\sum_{k=n}^{\infty} \Delta_k(w) = \sum_{k=n}^{\infty} 2^{-\alpha k} \cdot 2^{\alpha k} \Delta_k(w)$$

Proof of Claim:

Fix $n \in \mathbb{N}$. Expand $s = \frac{a_0}{2^n} + \sum_{k=1}^{\infty} \frac{a_k}{2^{n+k}}$ $\quad t = \frac{b_0}{2^n} + \sum_{k=1}^{\infty} \frac{b_k}{2^{n+k}}$

where $a_k, b_k \in \{0,1\}$, $a_k, b_k = 0$ $\forall$ large k. Since $t-s \leq 2^{-n}$,
$b_0 = a_0$ or $a_0 + 1$.

Set $s_m = \frac{a_0}{2^n} + \sum_{k=1}^{m} \frac{a_k}{2^{n+k}}$

$\therefore |s_m - s_{m+1}|$

$$s_m = \frac{a_0}{2^n} + \sum_{k=1}^{m} \frac{a_k}{2^{n+k}} \qquad t_m = \frac{b_0}{2^n} + \sum_{k=1}^{m} \frac{b_k}{2^{n+k}} \qquad \text{and} \quad 0 \le b_0 - a_0 \le 1$$

$$d(w(s_m), w(s_{m+1})) \le \Delta_{n+m+1}(w) \qquad d(t_m, t_{m+1}) \le \Delta_{n+m+1}(w)$$

$$\therefore d(w(s_0), w(s)) \qquad d(w(t_0), w(t))$$

$$\therefore d(w(s), w(t))$$

$$\le d(w(s), w(s_0)) + d(w(s_0), w(t_0)) + d(w(t_0), w(t))$$