Def: Let $w:[0,1]\to(S,d)$. Fix some $\alpha\in(0,1)$.
Say w is **Hölder-α continuous**, $w\in C^\alpha([0,1],S)$ if $\exists K=K(w)<\infty$ s.t. $\forall s,t\in[0,1]$, $d(w(s),w(t))\leq K(w)|s-t|^\alpha$.

Working with uncountable families of measurable sets is challenging.
Fortunately, Hölder continuity interacts well with (countable) dense subsets.

Lemma: Let $D\subset[0,1]$ be a dense subset. Let (S,d) be a complete metric space. Let $\alpha\in(0,1)$. If $w:D\to S$ is α-Höler continuous, then $\exists!\ \bar{w}:[0,1]\to S$ that is α-Hölder continuous (with $K(\bar{w})=K(w)$), and s.t. $\bar{w}|_D=w$.

Pf. Fix $t\in[0,1]$; we know $\exists\ (t_n)_{t\in\mathbb{N}}$ in D s.t. $t_n\to t$.
Define $\bar{w}(t):=\lim_{n\to\infty} w(t_n)$.

- Exists: $d(w(t_n),w(t_m))\leq K(w)\,|t_n-t_m|^\alpha \xrightarrow{n,m\to\infty} 0$
- well-defined: $t_n\to t,\ s_n\to t,\ d(w(t_n),w(s_n))\leq K\,|t_n-s_n|^\alpha\to 0$.
- $\bar{w}|_D=w$: if $t\in D$, $t_n\equiv t\ \forall n$. $\therefore\ \bar{w}(t)=\lim_{n\to\infty}w(t)=w(t)$.

• Hölder continuous: fix any sequences $t_n \to t$, $s_n \to s$.

$$d(\bar{\omega}(s), \bar{\omega}(t)) = d\left(\lim_{n\to\infty}(\omega(s_n), \omega(t_n))\right) = \lim_{n\to\infty} \underbrace{d(\omega(s_n), \omega(t_n))}_{\leq K(\omega)|s_n - t_n|^\alpha}$$

cont. on $S \times S$

$$\leq K(\omega) \limsup_{n\to\infty} |s_n - t_n|^\alpha = K(\omega)|s-t|^\alpha.$$

$\therefore K(\bar{\omega}) \leq K(\omega)$.

But $K(\omega) \leq K(\bar{\omega})$ ///

We're going to construct Hölder continuous versions by working on a countable dense subset of $[0,1]$.

Def: The dyadic rational numbers $\mathbb{D}$ are those real numbers whose binary expansion is finitely-terminating.

$$[0,\infty) \ni t = b_0(t) + \sum_{k=1}^{\infty} \frac{b_k(t)}{2^k} \quad \text{for unique } b_0(t) \in \mathbb{N}$$
$$b_k(t) \in \{0,1\},\ k \geq 1$$

$t \in \mathbb{D}$ iff $b_k(t) = 0$ $\forall$ suff. large k.

Notice: $\mathbb{D} = \bigcup_{n\in\mathbb{N}} \mathbb{D}_n$, where $\mathbb{D}_n = \{ \frac{i}{2^n}, i\in\mathbb{Z} \}$.

From this (interlacing) union, we get the following useful refinement of the characterization of $\mathbb{D}$:

For any $t\in[0,\infty)$, and any n, there are unique $a_0 = a_0^{(n)} \in \mathbb{N}$, $a_k = a_k^{(n)} \in\{0,1\}$ $(k\geq 1)$ s.t.

$$t = \frac{a_0}{2^n} + \sum_{k=1}^{\infty} \frac{a_k}{2^{n+k}}$$

(do binary expansion of $2^n t$.)

and $t\in\mathbb{D}_+ = \mathbb{D}\cap[0,\infty)$ iff $a_k=0$ for all large k.

In particular, a_0 is defined by $\frac{a_0}{2^n} \leq t < \frac{a_0+1}{2^n}$

$\{ [\frac{a}{2^n}, \frac{a+1}{2^n}) : a\in\mathbb{N}\}$ partitions $[0,\infty)$

Notice: if $w \in C^\gamma$ and $|s-t| \leq \varepsilon$, then $d(w(s), w(t)) \leq |s-t|^\gamma \leq K(w)\,\varepsilon^\gamma$

In particular, if $\varepsilon = 2^{-n}$, then $\leq K(w)\, 2^{-\gamma n}$.

Def: Let $w : \mathbb{D} \cap [0,1] \to (S,d)$. For each n, define

$$\Delta_n(w) := \max\{ d(w(s), w(t)) : s,t \in \underbrace{\mathbb{D}_n \cap [0,1]}_{\{\frac{i}{2^n} : 0 \leq i \leq 2^n\}},\ |s-t| \leq 2^{-n} \}$$

$s = \frac{i}{2^n}$, $t = \frac{i \pm 1}{2^n}$.

By the note above, if $w \in C^\gamma$, then $\Delta_n(w) \leq K(w)\, 2^{-\gamma n}$.

$\therefore$ for any $\alpha \in (0,\gamma)$,

$$\sum_{n=0}^{\infty} 2^{n\alpha} \Delta_n(w) \leq K(w) \sum_{n=0}^{\infty} \underbrace{2^{n\alpha}\, 2^{-n\gamma}}_{2^{-(\gamma-\alpha)n}} < \infty .$$

Prop: For each $\alpha \in (0,1)$, set $K_\alpha(w) = 2^{1+\alpha} \sum_{n=0}^{\infty} 2^{\alpha n} \Delta_n(w)$.

If $w : \mathbb{D} \cap [0,1] \to (S,d)$ satisfies $K_\alpha(w) < \infty$, then $w \in C^\alpha$, and

$$d(w(s), w(t)) \leq K_\alpha(w)\, |s-t|^\alpha \quad \forall s,t \in \mathbb{D} \cap [0,1].$$

Pf. The core of the proof is the following

Claim. If $s, t \in \mathbb{D} \cap [0,1]$ with $2^{-(n+1)} < t-s \leq 2^{-n}$, then

$$d(w(s), w(t)) \leq 2 \sum_{k=n}^{\infty} \Delta_k(w).$$

This will complete the proof, because

$$\sum_{k=n}^{\infty} \Delta_k(w) = \sum_{k=n}^{\infty} \underbrace{2^{-\alpha k}}_{\leq 2^{-\alpha n}} \cdot 2^{\alpha k} \Delta_k(w) \leq (2^{-n})^{\alpha} \sum_{k=n}^{\infty} 2^{\alpha k} \Delta_k(w) \leq 2^{\alpha} \sum_{k=n}^{\infty} 2^{\alpha k} \Delta_k(w) (t-s)^{\alpha}$$

$$2^{-n} = 2^{-(n+1)} \cdot 2 < 2(t-s)$$

$$\leq 2^{\alpha} \underbrace{\sum_{k=0}^{\infty} 2^{\alpha k} \Delta_k(w)}_{K_\alpha(w)/2} (t-s)^{\alpha}$$

Proof of Claim:

Fix $n \in \mathbb{N}$. Expand $s = \frac{a_0}{2^n} + \sum_{k=1}^{\infty} \frac{a_k}{2^{n+k}}$ $\quad t = \frac{b_0}{2^n} + \sum_{k=1}^{\infty} \frac{b_k}{2^{n+k}}$

where $a_k, b_k \in \{0,1\}$, $a_k, b_k = 0$ $\forall$ large k. Since $t-s \leq 2^{-n}$, $b_0 = a_0$ or $a_0 + 1$.

Set $s_m = \frac{a_0}{2^n} + \sum_{k=1}^{m} \frac{a_k}{2^{n+k}}$ $\quad s_m = s$ $\forall$ large m.

$\therefore$ $|s_m - s_{m+1}| = \frac{a_{m+1}}{2^{n+m+1}} \leq 2^{-(n+m+1)}$

$$d(w(s_m), w(s_{m+1})) \leq \Delta_{n+m+1}(w).$$

$s_m = \frac{a_0}{2^n} + \sum_{k=1}^{m} \frac{a_k}{2^{n+k}} = s$ $\quad t_m = \frac{b_0}{2^n} + \sum_{k=1}^{m} \frac{b_k}{2^{n+k}} = t$ $\quad \forall m \geq N.$ $\quad$ and $\quad 0 \leq b_0 - a_0 \leq 1$

$d(\omega(s_m), \omega(s_{m+1})) \leq \Delta_{n+m+1}(\omega)$ $\qquad d(t_m, t_{m+1}) \leq \Delta_{n+m+1}(\omega)$

$\therefore\ d(\omega(s_0), \omega(s)) \qquad\qquad d(\omega(t_0), \omega(t))$

$$\leq \sum_{m=0}^{\cancel{N-1}\ \infty} \underbrace{d(\omega(s_m), \omega(s_{m+1}))}_{\leq \Delta_{\underbrace{n+m+1}_{k}}(\omega)} \leq \sum_{k=n+1}^{\infty} \Delta_k(\omega)$$

$$\leq \sum_{k=n+1}^{\infty} \Delta_k(\omega).$$

$\therefore\ d(\omega(s), \omega(t))$

$$\leq d(\omega(s), \omega(s_0)) + \overbrace{d(\omega(s_0), \omega(t_0))} + d(\omega(t_0), \omega(t))$$

$$\leq \sum_{k=n+1}^{\infty} \Delta_k(\omega) \qquad |t_0 - s_0| = \frac{b_0 - a_0}{2^n} \leq \frac{1}{2^n} \qquad \leq \sum_{k=n+1}^{\infty} \Delta_k(\omega)$$

$$\leq \Delta_n(\omega)$$

$$\leq 2\Delta_n(\omega) + 2\sum_{k=n+1}^{\infty} \Delta_k(\omega) = 2\sum_{k=n}^{\infty} \Delta_k(\omega). \quad ///$$