Continuous Time Processes

From now on, we will be focusing on processes $(X_t)_{t\geq 0}$ $T = [0,\infty)$ or $[0, t_0]$

$X_t : (\Omega, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}) \to (S, \mathcal{B})$

The law of such a process is a measure on S^T

$$P_X(E) = \mathbb{P}\left(\omega\in\Omega : (t\mapsto X_t(\omega))_{t\in T} \in E\right)$$

what kinds of subsets E of path space can be measured?

I.e. what σ-field?

One choice: $(S^T, \mathcal{B}^{\otimes T}, P_X)$

Problem: this σ-field is too small.

↳ When $S = \mathbb{R}^d$, it doesn't contain

$\{$continuous paths$\}$

or $\{$right-continuous paths w left limits$\}$

Def: Let $X=(X_t)_{t\in T}$ and $Y=(Y_t)_{t\in T}$ be processes $(\Omega,\mathcal{F},\mathbb{P})\to(S,\mathcal{B})$.

Say Y is a version (or modification) of X if

$$\forall t\in T \quad X_t=Y_t \text{ a.s.}$$

Def: Say X and Y are indistinguishable if $\mathbb{P}^*(X\neq Y)=0$

I.e. $\exists N\in\mathcal{F}$, $\mathbb{P}(N)=0$, s.t.

$$\{X\neq Y\}\subseteq N$$

$$\mathbb{P}^*\{\exists t\in T\ X_t\neq Y_t\}=\mathbb{P}^*\Big(\bigcup_{t\in T}\{X_t\neq Y_t\}\Big)$$

Notice: if (S,d) is a metric space and $\mathcal{B}=\mathcal{B}(S,d)$, then

X,Y versions iff $0=\sup_{t\in T}\mathbb{P}(X_t\neq Y_t)$

X,Y indistinguishable iff $\quad 0=$

Eg. $(\Omega, \mathcal{F}, \mathbb{P}) = ([0,1], \mathcal{B}[0,1], \text{Unif})$

$X_t, Y_t : \Omega \to \mathbb{R}$

$X_t = 0 \qquad Y_t(\omega) = \mathbb{1}_{\{\omega = t\}}$.

For fixed t, $\mathbb{P}\{X_t \neq Y_t\} =$

But: $\{X \neq Y\} =$

Note: if $\{t_1, t_2, \ldots\}$ is a countable collection of times, and if $(Y_t)_{t\in T}$ is a version of $(X_t)_{t\in T}$, then

$\mathbb{P}(\exists n \in \mathbb{N} \text{ s.t. } X_{t_n} \neq Y_{t_n}) =$

↳ If T is countable, X, Y versions $\Leftrightarrow$ X, Y indistinguishable

↳ In general, if X, Y are versions, they have the [same finite-dimensional distributions].

When we constructed Markov processes, we did it essentially by specifying their finite-dimensional distributions:

$$\mathbb{P}^x \in \mathrm{Prob}(S^T, \mathcal{B}^{\otimes T}) \qquad \forall x \in S$$

I.e. we constructed the law of the process, realized on "path space" S^T.

Our goal now will be to find a continuous version of such processes, when possible.

I.e. given X, show $\exists$ version $\tilde{X}$ of X s.t. $(t \mapsto \tilde{X}_t(\omega))_{t \in T}$

Then $\tilde{X}$ will have the same f.d. distributions as X, so will be "the same" from the Markov process perspective. But now, instead of S^T, it lives on true path space

$$C(T, S) \subsetneq S^T$$

Actually, $C(T,S)$ is still too big for the methods we'll use.

We're going to construct versions that are somewhat more regular.

Def: Let $w:[0,\infty) \to (S,d)$. Fix some $\alpha \in (0,1)$

Say w is Hölder-α continuous, $w \in C^\alpha([0,\infty),S)$ if $\exists K = K_w < \infty$

s.t. $\forall s,t \in [0,\infty)$,

- If we were to take $\alpha = 1$, we get
- If we were to take $\alpha > 1$, we get
- If $0 < \alpha < \beta < 1$, and we restrict to $[0,T] \subset [0,\infty)$, then $C^\beta[0,T] \subseteq C^\alpha[0,T]$

We're going to present criteria (due to - who else?

), in terms of f.dim. distributions,

which guarantee that a process has a C^α version

for some $\alpha \in (0,1)$.