Continuous Time Processes

$T = [0,\infty)$ or $[0, t_0]$

From now on, we will be focusing on processes $(X_t)_{t\geq 0}$

$X_t : (\Omega, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}) \to (S, \mathcal{B})$

The law of such a process is a measure on S^T

$$P_X(E) = \mathbb{P}\left(\omega\in\Omega : (t\mapsto X_t(\omega))_{t\in T} \in E\right)$$

$E \subseteq S^T$

What kinds of subsets E of path space can be measured? I.e. what σ-field?

One choice: $(S^T, \mathcal{B}^{\otimes T}, P_X)$

$\mathcal{B}^{\otimes T} = \sigma(\pi_t : t\in T)$: $\pi_t : S^T \to S$, $\pi_t(\omega) = \omega(t)$

Problem: this σ-field is too small.

↳ When $S = \mathbb{R}^d$, it doesn't contain

{continuous paths}

or {right-continuous paths w left limits}

Def: Let $X=(X_t)_{t\in T}$ and $Y=(Y_t)_{t\in T}$ be processes $(\Omega,\mathcal{F},\mathbb{P})\to(S,\mathcal{B})$.

Say Y is a version (or modification) of X if

$$\forall t\in T \quad X_t=Y_t \text{ a.s.} \quad \text{i.e. } \mathbb{P}(X_t\neq Y_t)=0 \text{ for each } t\in T.$$

could have $\mathbb{P}(X_t=Y_t\ \forall t\in T)<1$ if T is uncountable.

Def: Say X and Y are indistinguishable if $\mathbb{P}^*(X\neq Y)=0$

$$\mathbb{P}^*\{\exists t\in T\ X_t\neq Y_t\}=\mathbb{P}^*\Big(\bigcup_{t\in T}\{X_t\neq Y_t\}\Big)$$

I.e. $\exists N\in\mathcal{F}$, $\mathbb{P}(N)=0$, s.t.

$$\{X\neq Y\}\subseteq N$$

For each $t\in T$, $\{X_t\neq Y_t\}\subset\{X\neq Y\}\subseteq N$

$$\therefore \mathbb{P}(X_t\neq Y_t)\leq\mathbb{P}(N)=0.$$

Notice: if (S,d) is a metric space and $\mathcal{B}=\mathcal{B}(S,d)$, then

X,Y versions iff $0=\sup_{t\in T}\mathbb{P}(X_t\neq Y_t)=\sup_{t\in T}\mathbb{P}(d(X_t,Y_t)>0)$

X,Y indistinguishable iff $0=\mathbb{P}^*\{\sup_{t\in T}d(X_t,Y_t)>0\}$

Eg. $(\Omega, \mathcal{F}, \mathbb{P}) = ([0,1], \mathcal{B}[0,1], \text{Unif})$

$X_t, Y_t : \Omega \to \mathbb{R}$

$X_t = 0 \qquad Y_t(\omega) = \mathbb{1}_{\{\omega = t\}}$.

$\therefore X_t, Y_t$ versions.

For fixed t, $\mathbb{P}\{X_t \neq Y_t\} = \mathbb{P}(Y_t \neq 0) = \mathbb{P}(\omega : \omega = t) = 0$.

But: $\{X \neq Y\} = \{\exists t \in [0,1] : X_t(\omega) \neq Y_t(\omega)\}$

$= \{\omega : \exists t \in [0,1] \ \mathbb{1}\{\omega = t\} \neq 0\} = \Omega$

$\therefore \mathbb{P}(X \neq Y) = \mathbb{P}(\Omega) = 1$.

Not indistinguishable.

Note: if $\{t_1, t_2, \dots\}$ is a countable collection of times, and if $(Y_t)_{t\in T}$ is a version of $(X_t)_{t \in T}$, then

$$\mathbb{P}(\exists n \in \mathbb{N} \text{ s.t. } X_{t_n} \neq Y_{t_n}) = \mathbb{P}\Big(\bigcup_n \{X_{t_n} \neq Y_{t_n}\}\Big)$$
$$\leq \sum_n \mathbb{P}(X_{t_n} \neq Y_{t_n}) = 0.$$

↳ If T is countable, X, Y versions $\Leftrightarrow$ X, Y indistinguishable

↳ In general, if X, Y are versions, they have the [same finite-dimensional distributions]

When we constructed Markov processes, we did it essentially by specifying their finite-dimensional distributions:

$$\mathbb{P}^x \in \mathrm{Prob}(S^T, \mathcal{B}^{\otimes T}) \qquad \forall x \in S$$

I.e. we constructed the law of the process, realized on "path space" S^T.

Our goal now will be to find a continuous version of such processes, when possible.

I.e. given X, show $\exists$ version $\tilde{X}$ of X s.t. $(t \mapsto \tilde{X}_t(\omega))_{t \in T} \in C(T, S)$.

Then $\tilde{X}$ will have the same f.d. distributions as X, so will be "the same" from the Markov process perspective. But now, instead of S^T, it lives on true path space

$$C(T, S) \subsetneq S^T$$

Actually, $C(T,S)$ is still too big for the methods we'll use.
We're going to construct versions that are somewhat more regular.

Def: Let $\omega : [0,\infty) \to (S,d)$ $\omega \in S^{[0,\infty)}$. Fix some $\alpha \in (0,1)$.
Say ω is Hölder-α continuous, $\omega \in C^\alpha([0,\infty), S)$ if $\exists K = K_\omega < \infty$
s.t. $\forall s,t \in [0,\infty)$, $d(\omega(s), \omega(t)) \leq K_\omega |s-t|^\alpha$

- If we were to take $\alpha = 1$, we get Lipschitz functions. $\supsetneq C^1$ $(S = \mathbb{R}^d)$
- If we were to take $\alpha > 1$, we get constant functions [HW].
- If $0 < \alpha < \beta < 1$, and we restrict to $[0,T] \subset [0,\infty)$, then $C^\beta[0,T] \subseteq C^\alpha[0,T]$

$$|s-t|^\beta = |s-t|^\alpha \cdot |s-t|^{\beta-\alpha} \leq T^{\beta-\alpha} |s-t|^\alpha.$$

We're going to present criteria (due to – who else? Kolmogorov), in terms of f.dim. distributions, which guarantee that a process has a C^α version for some $\alpha \in (0,1)$.