Upcrossings are a measure of oscillation.

↳ Given a sequence $X = (X_n)_{n\in\mathbb{N}}$ in $\overline{\mathbb{R}}$, $U_N^X(a,b)$ = # times X crosses $[a,b]$ upwards in $(X_0, \dots, X_N)$

- If $X_n \uparrow$, $U_N^X(a,b) \leq 1$
- If $X_n \downarrow$, $U_N^X(a,b) = 0$. $\forall N$.

Suppose $\limsup_{n\to\infty} X_n \neq \liminf_{n\to\infty} X_n$. $\therefore \liminf_{n\to\infty} X_n < a < b < \limsup_{n\to\infty} X_n$ $a,b \in \mathbb{Q}$.

$$U_\infty^N(a,b) := \lim_{N\to\infty} U_N^X(a,b) = \infty.$$

Doob's Upcrossing Inequality: if X is a submartingale,

$$\mathbb{E}[U_N^X(a,b)] \leq \frac{1}{b-a}\left(\mathbb{E}[(X_N-a)_+] - \mathbb{E}[(X_0-a)_+]\right) \quad \forall N,\ a<b$$

Theorem: If X is an L^1-bounded submartingale, then $\lim_{n\to\infty} X_n =: X_\infty$ exists in $\mathbb{R}$ a.s., and $X_\infty \in L^1$.

Note: suffices just to assume $\sup_n \mathbb{E}[X_n^+] < \infty$

$$\mathbb{E}|X_n| = \mathbb{E}[X_n^+ + X_n^-] = \mathbb{E}[2X_n^+ - (X_n^+ - X_-)]$$
$$= 2\mathbb{E}[X_n^+] - \underbrace{\mathbb{E}[X_n]}_{\leq -\mathbb{E}[X_0]}.$$

Pf. For any $a<b$, $\mathbb{E}[U_N^X(a,b)] \le \frac{1}{b-a}\Big(\underbrace{\mathbb{E}[(X_N-a)_+] - \mathbb{E}[(X_0-a)_+]}\Big)$

$U_N^X(a,b) \uparrow U_\infty^X(a,b)$ $\qquad \le \mathbb{E}|X_N| + \mathbb{E}|X_0| + 2|a|$

$\therefore \mathbb{E}[U_\infty^X(a,b)] = \lim_{N\to\infty} \mathbb{E}[U_N^X(a,b)] \le \sup_k \mathbb{E}|X_k| + \mathbb{E}[|X_0|] + 2|a| \quad \forall N.$

$< \infty.$

So, if $\Omega_{a,b} = \{\omega : U_\infty^{X(\omega)}(a,b) < \infty\}$ then $\mathbb{P}(\Omega_{a,b}) = 1$.

$\therefore \Omega_0 := \bigcap_{\substack{a<b \\ a,b\in\mathbb{Q}}} \Omega_{a,b}$ has $\mathbb{P}=1$. But on Ω_0, $X_\infty := \lim_{n\to\infty} X_n \in \overline{\mathbb{R}}$.

Must show $X_\infty \in L^1$

$\mathbb{E}[|X_\infty|] = \mathbb{E}[\liminf_{n\to\infty} |X_n|] \le \liminf_{n\to\infty} \mathbb{E}[|X_n|]$

$\le \sup_n \mathbb{E}[|X_n|] < \infty.$

$\therefore X_\infty \in L^1$, $\therefore X_\infty \in \mathbb{R}$ a.s. ///

Note: $\lim_{n\to\infty} X_n = X_\infty$ a.s. and $X_\infty \in L^1$

$\not\Rightarrow \|X_n - X_\infty\|_{L^1} \to 0$

happens iff $\{X_n\}_{n\in\mathbb{N}}$ is UI.

Earlier, we saw that regular martingales $X_n = \mathbb{E}[X|\mathcal{F}_n]$ are UI.
It turns out that, in the L^1-bounded category, the converse is true.

Theorem: Let $(X_n)_{n\in\mathbb{N}}$ be an L^1-bounded (sub)martingale; let $X_\infty := \lim_{n\to\infty} X_n$.
Then $\{X_n\}_{n\in\mathbb{N}}$ is UI iff $X_n \underset{(\le)}{=} \mathbb{E}[X_\infty|\mathcal{F}_n]$ $\forall n$.

Pf. (martingale case)

($\Longrightarrow$) By the Vitali convergence theorem, $X_n \to X_\infty$ in L^1.
Fix n; for $m \ge n$, $X_n = \mathbb{E}[X_m|\mathcal{F}_n] \longrightarrow \mathbb{E}[X_\infty|\mathcal{F}_n]$

$$\|\mathbb{E}_{\mathcal{F}_n}[X_m] - \mathbb{E}_{\mathcal{F}_n}[X_\infty]\|_{L^1} = \|\mathbb{E}_{\mathcal{F}_n}[X_m - X_\infty]\|_{L^1} \le \|X_m - X_\infty\|_{L^1} \xrightarrow[m\to\infty]{} 0.$$

($\Longleftarrow$) If $X_n = \mathbb{E}[X_\infty|\mathcal{F}_n]$ where $X_\infty \in L^1$, then
$(X_n)_{n\in\mathbb{N}}$ is a regular martingale. $\therefore$ From [Lec 48.1]
we know $\{X_n\}_{n\in\mathbb{N}}$ is UI. ///

(For the submartingale case, see [Driver, Cor 23.59].)

Cor: Let $1<p<\infty$. Suppose $(X_n)_{n\in\mathbb{N}}$ is an L^p-bounded martingale.

Then $\lim_{n\to\infty} X_n =: X_\infty$ exists a.s., $X_\infty \in L^p$, and $\|X_n - X_\infty\|_{L^p} \to 0$.

In particular, $(X_n)_{n\in\mathbb{N}}$ is a regular martingale: $X_n = \mathbb{E}[X_\infty | \mathcal{F}_n]$ a.s.

Pf. Let $Y_n = |X_n|^p$. Since $(X_n)_{n\in\mathbb{N}}$ is a martingale, Y_n is a submartingale.

$\|Y_n\|_{L^1} = \mathbb{E}[|X_n|^p] = \|X_n\|_{L^p}^p \quad \therefore \sup_n \|Y_n\|_{L^1} = \sup_n \|X_n\|_{L^p}^p < \infty$.

$\therefore Y_\infty := \lim_{n\to\infty} Y_n$ exists a.s. and is in L^1.

Also: $\|X_n\|_{L^1} = \|X_n \cdot 1\|_{L^1} \le \|X_n\|_{L^p} \|\underbrace{1}_{1}\|_{L^{p'}} = \|X_n\|_{L^p} \quad \therefore (X_n)$ is L^1-bounded.

$\therefore X_\infty = \lim_{n\to\infty} X_n$ exists a.s.

$L^1 \ni Y_\infty = \lim_{n\to\infty} Y_n = \lim_{n\to\infty} |X_n|^p = |X_\infty|^p \Rightarrow X_\infty \in L^p$.

Now, to show $\|X_n - X_\infty\|_{L^p} \to 0$, by Vitali, suffices to show $\{|X_n|^p\}_{n\in\mathbb{N}}$ is UI.

Claim: $\{Y_n\}_{n\in\mathbb{N}} = \{|X_n|^p\}_{n\in\mathbb{N}}$ is UI.

Suffices to show $\{Y_n\}_{n\in\mathbb{N}}$ has a uniform dominating function $g \in L^1$ [Lec 48.1]

$$Y_n = |X_n|^p \leq \sup_m |X_m|^p =: g.$$

$$\mathbb{E}[g] = \mathbb{E}\left[\sup_m |X_m|^p\right]$$

$$= \mathbb{E}\left[\lim_{N\to\infty} \sup_{m\leq N} |X_m|^p\right] \leq \lim_{N\to\infty} \mathbb{E}\left[\sup_{m\leq N} |X_m|^p\right]$$

$$\mathbb{E}\left[\sup_{m\leq N} |X_m|^p\right] \leq (p')^p\, \mathbb{E}[|X_N|^p]$$

$$\therefore \mathbb{E}[g] \leq \limsup_{N\to\infty} (p')^p\, \mathbb{E}[|X_N|^p] = (p')^p \|X_\infty\|_{L^p}^p < \infty.$$

Thus, $\{Y_n\}_{n\in\mathbb{N}}$ is UI, and $\therefore$ by Vitali, $X_n \to X_\infty$ in L^p.

Finally, $\{X_n\}_{n\geq 1}^{\infty}$ is L^p-bounded for some $p > 1$,

so it is UI; also, it is L^1-bounded

$\therefore$ By the last Theorem, $X_n = \mathbb{E}[X_\infty \mid \mathcal{F}_n]$ a.s. ///

As a final note: recall the Optional Sampling Theorem, which in general requires bounded stopping times.

Theorem: (Optional Sampling Theorem, II)

Let $(X_n)_{n\in\mathbb{N}}$ be a regular martingale, and let $X_\infty := \lim_{n\to\infty} X_n$.
Then for any two stopping times σ, τ:

$$X_\tau = \mathbb{E}[X_\infty | \mathcal{F}_\tau], \quad \mathbb{E}[|X_\tau|] \le \mathbb{E}[|X_\infty|] < \infty, \text{ and}$$

$$\mathbb{E}[X_\tau | \mathcal{F}_\sigma] = X_{\tau\wedge\sigma}.$$

Pf. Since (X_n) is regular, it's UI; $\therefore$ by the last Theorem, $X_n = \mathbb{E}[X_\infty | \mathcal{F}_n]$. It follows that $X_\tau = \mathbb{E}[X_\infty | \mathcal{F}_\tau]$ [Lec 45.3]

$\therefore \mathbb{E}[|X_\tau|] = \mathbb{E}[|\mathbb{E}_{\mathcal{F}_\tau}[X_\infty]|] \le \mathbb{E}[\mathbb{E}_{\mathcal{F}_\tau}[|X_\infty|]] = \|X_\infty\|_{L^1} < \infty$.

Finally, the general tower property [Lec 45.3], as $X_\infty \in L^1$,

$$\mathbb{E}_{\mathcal{F}_\sigma}[X_\tau] = \mathbb{E}_{\mathcal{F}_\sigma}\mathbb{E}_{\mathcal{F}_\tau}[X_\infty] = \mathbb{E}_{\mathcal{F}_{\sigma\wedge\tau}}[X_\infty] = X_{\sigma\wedge\tau}.$$

///