Back to the Stock Market.

$(X_n)_{n\in\mathbb{N}}$ models a stock price. We assume it is a submartingale $(\Omega, \{\mathcal{F}_n\}_{n\in\mathbb{N}}, \mathbb{P})$

You (investor) have an initial fortune W_0, and then by some amount A_n of the stock at time $n-1$; so $\{A_n\}_{n=1}^{\infty}$ is a predictable process.

Your fortune at time n is

$$W_n = W_0 + I_n(A, X)$$

Question: What's your best betting strategy? Buy low, sell high?

Surprisingly, NO.

↳ There's a daily limit on how much you can buy.

Let's normalize to make the limit 1:

It turns out the optimal strategy is:

Prop: If $(X_n)_{n\geq 0}$ is a submartingale, $(A_n)_{n\geq 1}$ is predictable, $A_n \in [0,1]$ $\forall n$.
then
$$\mathbb{E}\left[\sum_{j=1}^{n} A_j (X_j - X_{j-1})\right]$$

Pf: Set $C_n = 1 - A_n$

Since $(X_n)_{n\geq 0}$ is a submartingale, $I_n(C,X)$ is a submartingale.

$$X_n - X_0 = \sum_{j=1}^{n} (X_j - X_{j-1})$$

Buy Low, Sell High

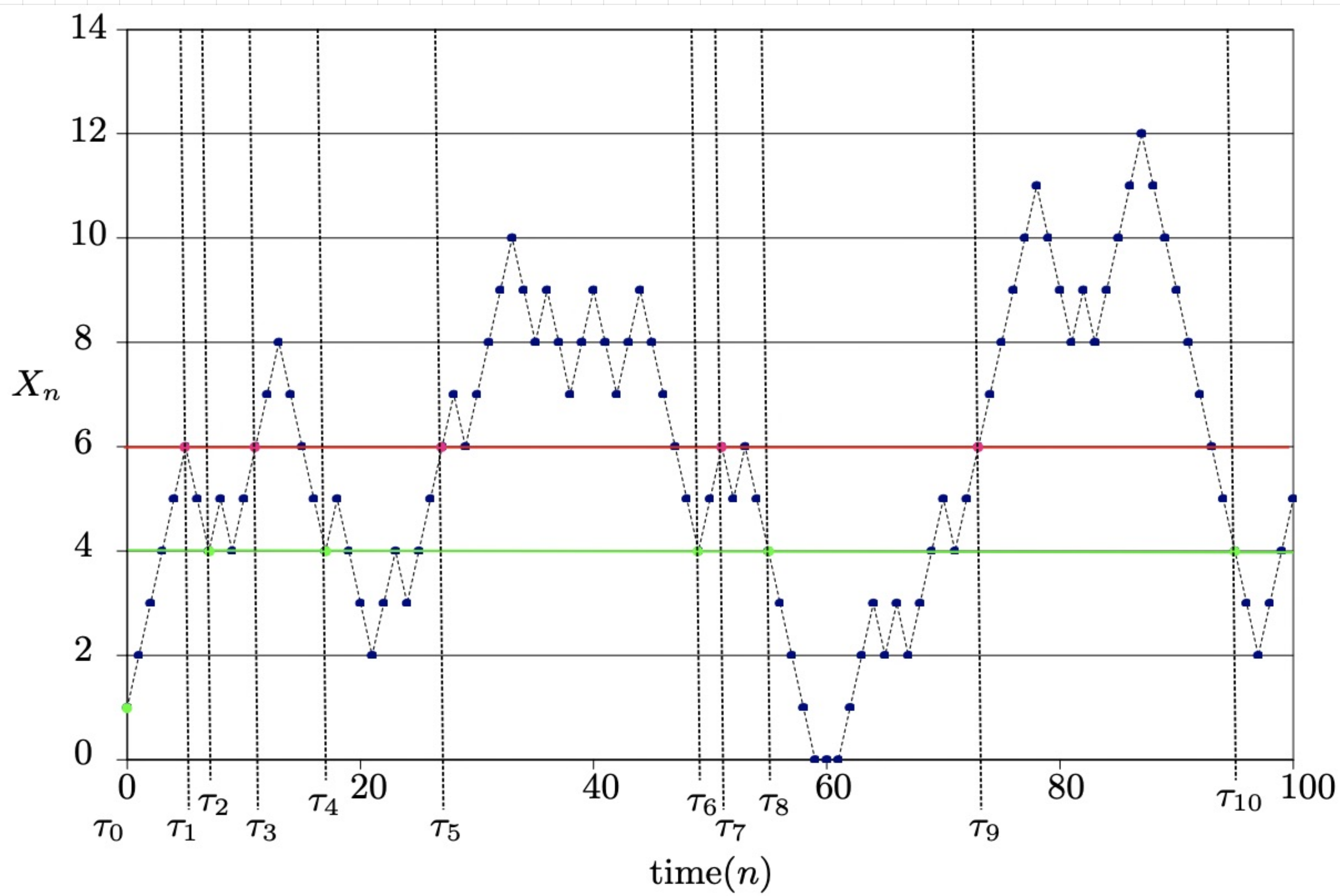
Fix $a < b$.

- wait to buy until the prize $X_n \leq a$
- wait to sell until the prize $X_n \geq b$

$\tau_0 = \inf\{n \geq 0 : X_n \leq a\}$ $\qquad$ $\tau_1 = \inf\{n \geq \tau_0 : X_n \geq b\}$

Betting Strategy:

$A_n =$



Def: For each $N \in \mathbb{N}$, let $U_N^X(a,b)$ = # times X crosses $[a,b]$ upward

The use of the buy-low, sell-high strategy comes from the fact that each upcrossing contributes $b-a$ profit to the investor. I.e. even if the price drops lower than b after we buy, that loss is cancelled during the next upturn, en route to an upcrossing.

$$W_N = \sum_{j=1}^{N} A_j \Delta X_j \approx (b-a) U_N^X(a,b)$$

- It could be more: if the price is actually $< a$ when we buy
- It could be less: if the price stays $< a$ around time N.

Precise lower bound:

W_N

Theorem: (Doob's Upcrossing Inequality)

If $(X_n)_{n\geq 0}^{\infty}$ is a submartingale and $-\infty < a < b < \infty$, then for $N \in \mathbb{N}$

$$\mathbb{E}[U_N^X(a,b)] \leq \frac{1}{b-a}\left(\mathbb{E}[(X_N-a)_+] - \mathbb{E}[(X_0-a)_+]\right)$$

Pf. Let $(A_n)_{n=1}^{\infty}$ denote the buy-low sell-high strategy for $a<b$.

Let $W_N = \sum_{j=1}^{N} A_j \Delta X_j$. We know that $\mathbb{E}[W_N] \leq \mathbb{E}[X_N - X_0]$.

I.e. $\mathbb{E}[(X_N-a) - (X_0-a)] = \mathbb{E}[X_N - X_0]$

$$\geq \mathbb{E}[W_N]$$

Thus $(b-a)\,\mathbb{E}[U_N^X(a,b)]$

$$\leq \mathbb{E}[(X_N-a)_+ + (X_N-a)_-] - \mathbb{E}[(X_0-a)_+ + (X_0-a)_-]$$

Note: $(X_n-a)_+ = \max(X_n-a, 0)$ is a submartingale.