Back to the Stock Market.

$(X_n)_{n\in\mathbb{N}}$ models a stock price. We assume it is a submartingale $(\Omega, \{\mathcal{F}_n\}_{n\in\mathbb{N}}, \mathbb{P})$

You (investor) have an initial fortune W_0, and then by some amount A_n of the stock at time $n-1$; so $\{A_n\}_{n=1}^{\infty}$ is a predictable process.

Your fortune at time n is

$$W_n = W_0 + I_n(A, X) = W_0 + \sum_{j=1}^{n} A_j (X_j - X_{j-1})$$

Question: What's your best betting strategy? Buy low, sell high?

Surprisingly, NO.

↳ There's a daily limit on how much you can buy.

Let's normalize to make the limit 1: $A_n \in [0,1]$

It turns out the optimal strategy is:

ALL IN! $A_n = 1 \ \forall n$.

Prop: If $(X_n)_{n\geq 0}$ is a submartingale, $(A_n)_{n\geq 1}$ is predictable, $A_n \in [0,1]$ $\forall n$.

then
$$\mathbb{E}\left[\sum_{j=1}^{n} A_j (X_j - X_{j-1})\right] \leq \mathbb{E}[X_n - X_0] \quad \forall n.$$

Pf. Set $C_n = 1 - A_n \geq 0$, still predictable.

Since $(X_n)_{n\geq 0}$ is a submartingale, $I_n(C,X)$ is a submartingale.

$$\mathbb{E}\left[\sum_{j=1}^{n} C_j \Delta X_j\right] = \mathbb{E}[I_n(C,X)] \geq 0.$$

$$X_n - X_0 = \sum_{j=1}^{n} (X_j - X_{j-1}) = \sum_{j=1}^{n} C_j \Delta X_j + \sum_{j=1}^{n} A_j \Delta X_j.$$

$1 = A_j + C_j$

$$\mathbb{E}[X_n - X_0] = \underbrace{\mathbb{E}[I_n(C,X)]}_{\geq 0} + \mathbb{E}[I_n(A,X)]$$

$$\geq \mathbb{E}\left[\sum_{j=1}^{n} A_j (X_j - X_{j-1})\right]$$

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Buy Low, Sell High

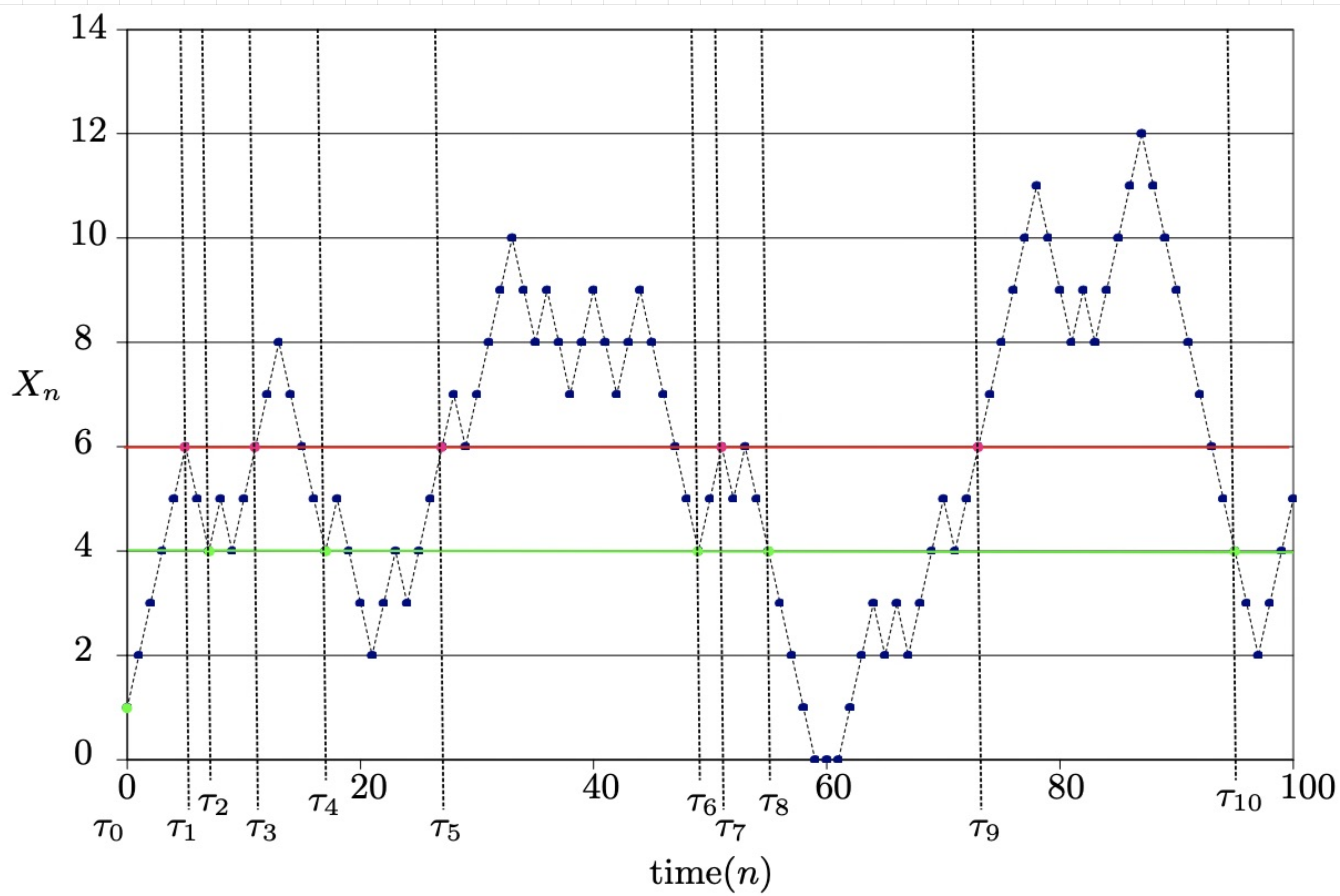
Fix $a < b$.

- wait to buy until the prize $X_n \leq a$
- wait to sell until the prize $X_n \geq b$

} repeat.

$\tau_0 = \inf\{n \geq 0 : X_n \leq a\}$ $\quad$ $\tau_1 = \inf\{n \geq \tau_0 : X_n \geq b\}$

$\tau_2 = \inf\{n \geq \tau_1 : X_n \leq a\}$ $\quad$ $\tau_3 = \inf\{n \geq \tau_2 : X_n \geq b\}$ $\vdots$



Betting Strategy:

$$A_n = \sum_{k=0}^{\infty} \mathbb{1}_{\tau_{2k} < n \leq \tau_{2k+1}}$$

$$\mathbb{1}_{\{\tau_{2k} < n\}} \mathbb{1}_{\{\tau_{2k+1} \geq n\}}$$

$\{\tau_{2k} < n\} = \{\tau_{2k} \leq n-1\} \in \mathcal{F}_{n-1}$

$\{\tau_{2k+1} \geq n\} = \{\tau_{2k} < n\}^c \in \mathcal{F}_{n-1}$

$\therefore (A_n)_n$ predictable.

Def: For each $N \in \mathbb{N}$, let $U_N^X(a,b) = \#$ times X crosses $[a,b]$ upward
$= \inf\{k \geq 1 : \tau_{2k-1} \leq N\}$

The use of the buy-low, sell-high strategy comes from the fact that each upcrossing contributes $b-a$ profit to the investor. I.e. even if the prize drops lower than b after we buy, that loss is cancelled during the next upturn, en route to an upcrossing.

$$W_N = \sum_{j=1}^{N} A_j \Delta X_j \approx (b-a)\, U_N^X(a,b)$$

- It could be more: if the price is actually $<a$ when we buy
- It could be less: if the price stays $<a$ around time N.

Precise lower bound:

$$W_N \geq (b-a)\, U_N^X(a,b) + (a-X_0)\mathbb{1}_{X_0 \leq a} - (a-X_N)\mathbb{1}_{X_N \leq a}$$
$$= (b-a)\, U_N^X(a,b) + (X_0-a)_- - (X_N-a)_-.$$

Theorem: (Doob's Upcrossing Inequality)

If $(X_n)_{n\geq 0}$ is a submartingale and $-\infty < a < b < \infty$, then for $N \in \mathbb{N}$

$$\mathbb{E}[U_N^X(a,b)] \leq \frac{1}{b-a}\left(\mathbb{E}[(X_N-a)_+] - \mathbb{E}[(X_0-a)_+]\right)$$

Pf. Let $(A_n)_{n=1}^{\infty}$ denote the buy-low sell-high strategy for $a<b$.

Let $W_N = \sum_{j=1}^{N} A_j \Delta X_j$. We know that $\mathbb{E}[W_N] \leq \mathbb{E}[X_N - X_0]$.

I.e. $\mathbb{E}[(X_N-a)-(X_0-a)] = \mathbb{E}[X_N - X_0]$

$$\geq \mathbb{E}[W_N] \geq \mathbb{E}[(b-a)U_N^X(a,b) + (X_0-a)_- - (X_N-a)_-].$$

Thus $(b-a)\,\mathbb{E}[U_N^X(a,b)]$

$$\leq \mathbb{E}[\underbrace{(X_N-a)+(X_N-a)_-}_{(X_N-a)_+}] - \mathbb{E}[\underbrace{(X_0-a)+(X_0-a)_-}_{(X_0-a)_+}] \qquad ///$$

$f = f_+ - f_-$

Note: $(X_n-a)_+ = \max(X_n-a, 0)$ is a submartingale.

$g(x) = \max(x-a, 0)$ convex, $\uparrow$