Def: Let $\{X_n\}_{n\in\mathbb{N}}$ be any sequence in $\mathbb{R}\cup\{\pm\infty\}$. Set

$$X_n^* := \max\{X_0, X_1, \dots, X_n\}$$

Estimates for the running max are often powerful tools.

E.g. In [Lec. 18.2] we proved and used Kolmogorov's maximal inequality:
if $\{\xi_n\}_{n=1}^{\infty}$ are independent, centered, L^2 random variables, and

$$X_n = |\xi_1 + \cdots + \xi_n|$$

then $$\mathbb{P}(X_n^* \geq \varepsilon) \leq \frac{1}{\varepsilon^2}\mathbb{E}[X_n^2 : X_n^* \geq \varepsilon]$$

(This was the key to proving Kolmogorov's Convergence criterion, and $\therefore$ SLLN.
You should review the proof: you'll see that it really used stopping times!)

We'll see that Kolmogorov's maximal inequality is really a special case of more general, powerful bounds for submartingales.

Prop. (Lévy's maximal inequality) $(\Omega, \{\mathcal{F}_n\}_{n\geq 0}, \mathbb{P})$

Let $(X_n)_{n\in\mathbb{N}}$ be a submartingale, and let $N\in\mathbb{N}$. For any $a\geq 0$,

$$\mathbb{P}(X_N^* \geq a) \leq \frac{1}{a}\mathbb{E}[X_N : X_N^* \geq a]$$

Pf. Set $\tau = \tau_{[a,\infty]} = \inf\{n\geq 0 : X_n \geq a\}$, a stopping time.

By the Optional Sampling Theorem,

$$\mathbb{E}[X_N \mid \mathcal{F}_{\tau\wedge N}] \geq X_{\tau\wedge N}$$

$$\therefore \mathbb{E}[X_\tau : \tau \leq N] = \mathbb{E}[X_{\tau\wedge N} : \tau \leq N]$$

Note that $\{\tau \leq N\} = \{X_n \geq a$ for some $n \leq N\}$

$\mathbb{E}[X_N : X_N^* \geq a] \geq \mathbb{E}[X_\tau : \tau \leq N]$ where $\tau = \inf\{n \geq 0 : X_n \geq a\}$

$\therefore X_\tau \geq a$ on $\{\tau < \infty\}$

We're going to use this to prove an L^p martingale inequality for $p > 1$.

Theorem: (Doob's Submartingale Inequality)
Let $(X_n)_{n\geq 0}$ be a ≥ 0 submartingale, and let $p > 1$. Then $\forall N \in \mathbb{N}$,

$$\mathbb{E}[(X_N^*)^p]^{1/p} = \| X_N^* \|_{L^p} \leq p' \| X_N \|_{L^p}$$

Cor: (Doob's Martingale Inequality)
If $(M_n)_{n\geq 0}$ is a martingale, $\mathbb{P}(|M|_N^* \geq a) \leq \frac{1}{a}\mathbb{E}[|M_N| : M_N^* \geq a] \leq \frac{1}{a}\mathbb{E}[|M_N|]$
and if $M_n \in L^p$ for some $p > 1$ then $\| |M|_N^* \|_{L^p} \leq p' \| M_N \|_{L^p}$.

Pf. 1. Apply Lévy
2. Apply Doob
to the submartingale $X_n = |M_n|$.

Lemma: Let X, Y be ≥ 0 rv's with $\mathbb{P}(Y \geq y) \leq \frac{1}{y} \mathbb{E}[X : Y \geq y]$ $\forall y > 0$.

Let $p > 1$. Assume $Y \in L^p$. Then $\mathbb{E}[Y^p] \leq (p')^p \, \mathbb{E}[X^p]$.

Pf. $Y^p - 0^p = \int_0^Y p y^{p-1} \, dy = p \int_0^\infty \mathbb{1}\{y \leq Y\} \, y^{p-1} \, dy$

By Tonelli's theorem, $\mathbb{E}[Y^p] = p \int_0^\infty \mathbb{E}[\mathbb{1}_{\{y \leq Y\}}] \, y^{p-1} \, dy$

Now we use Hölder's inequality.

$\mathbb{E}[X Y^{p-1}] \leq \|X\|_{L^p} \, \|Y^{p-1}\|_{L^{p'}}$

So $\|Y\|_{L^p}^p \leq p' \, \|X\|_{L^p} \, \|Y\|_{L^p}^{\cancel{p-1}}$.

Theorem: (Doob's Submartingale Inequality)

Let $(X_n)_{n\geq 0}$ be a ≥ 0 submartingale, and let $p>1$. Then $\forall N \in \mathbb{N}$,

$$\mathbb{E}[(X_N^*)^p] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}[(X_N)^p]$$

Pf. Since $X_n = |X_n|$ is a submartingale, by Lévy's maximal inequality,

$$\mathbb{P}(X_N^* \geq a) \leq \frac{1}{a}\mathbb{E}[X_N ; X_N^* \geq a]$$

$\therefore$ By Lemma,

Sidenote: there is no L^1 (sub) martingale inequality.

Eg. S_n = symmetric random walk starting at 1.

$X_n = S_n^{\tau_0} \leftarrow$ martingale (by optional stopping),

$\mathbb{E}[X_n] =$

What can we say about $\mathbb{E}[X_n^*]$?

We proved [Lec 43.2] $\mathbb{P}^1(\tau_a < \tau_0) = \frac{1}{a-0} \quad \forall a > 0$.

$$\mathbb{E}\left[\max_{m \geq 0} X_m\right] = \sum_{a=1}^{\infty} \mathbb{P}\left(\max_{m \geq 0} X_m \geq a\right) \qquad \text{(See [Lec 46.3])}$$

$$= \infty.$$

But $X_n^* = \max\{X_0, \ldots, X_n\} \uparrow \max_{m \geq 0} X_m$, so by MCT, $\mathbb{E}[X_n^*] \uparrow \infty$

while $\mathbb{E}[X_n] = 1 \quad \forall n$. !! ☹