Def: Let $\{X_n\}_{n\in\mathbb{N}}$ be any sequence in $\mathbb{R}\cup\{\pm\infty\}$. Set

$$X_n^* := \max\{X_0, X_1, \ldots, X_n\}$$

Estimates for the running max are often powerful tools.

E.g. In [Lec. 18.2] we proved and used Kolmogorov's maximal inequality:

if $\{\xi_n\}_{n=1}^{\infty}$ are independent, centered, L^2 random variables, and

$$X_n = |\xi_1 + \cdots + \xi_n|$$

then $$\mathbb{P}(X_n^* \geq \varepsilon) \leq \frac{1}{\varepsilon^2}\mathbb{E}[X_n^2 : X_n^* \geq \varepsilon] \leq \frac{1}{\varepsilon^2}\mathbb{E}[X_n^2]$$

(This was the key to proving Kolmogorov's Convergence criterion, and $\therefore$ SLLN. You should review the proof: you'll see that it really used stopping times!)

We'll see that Kolmogorov's maximal inequality is really a special case of more general, powerful bounds for submartingales.

Prop (Lévy's maximal inequality) $(\Omega, \{\mathcal{F}_n\}_{n\geq 0}, \mathbb{P})$

Let $(X_n)_{n\in\mathbb{N}}$ be a submartingale, and let $N\in\mathbb{N}$. For any $a\geq 0$,

$$\mathbb{P}(X_N^* \geq a) \leq \frac{1}{a}\mathbb{E}[\underbrace{X_N}_{\leq X_N^+} : X_N^* \geq a] \leq \frac{1}{a}\mathbb{E}[X_N^+]$$

Pf. Set $\tau = \tau_{[a,\infty)} = \inf\{n\geq 0 : X_n \geq a\}$, a stopping time.

By the Optional Sampling Theorem,

$$\mathbb{E}[X_N \mid \mathcal{F}_{\tau\wedge N}] \geq X_{\tau\wedge N}$$

$\{\tau\leq N\} \in \mathcal{F}_{\tau\wedge N}$

$$\therefore \mathbb{E}[X_\tau : \tau\leq N] = \mathbb{E}[X_{\tau\wedge N} : \tau\leq N]$$

$$\leq \mathbb{E}[\mathbb{E}_{\mathcal{F}_{\tau\wedge N}}[X_N]\mathbb{1}_{\tau\leq N}]$$

$$= \mathbb{E}[X_N \mathbb{1}_{\{\tau\leq N\}}] = \mathbb{E}[X_N : X_N^* \geq a].$$

Note that $\{\tau\leq N\} = \{X_n \geq a \text{ for some } n\leq N\}$

$$= \{X_N^* \geq a\}$$

$$\mathbb{E}[X_\tau : \tau\leq N] \leq \mathbb{E}[X_N : X_N^* \geq a].$$

$$\mathbb{E}[X_N : X_N^* \geq a] \geq \mathbb{E}[X_\tau : \tau \leq N] \quad \text{where } \tau = \inf\{n \geq 0 : X_n \geq a\}$$

$$\geq \mathbb{E}[a : \tau \leq N] \qquad \because X_\tau \geq a \text{ on } \{\tau < \infty\}$$

$$= a\,\mathbb{P}(\tau \leq N)$$

$$= a\,\mathbb{P}(X_N^* \geq a) \qquad ///$$

We're going to use this to prove an L^p martingale inequality for $p > 1$.

Theorem: (Doob's Submartingale Inequality)
Let $(X_n)_{n \geq 0}$ be a ≥ 0 submartingale, and let $p > 1$. Then $\forall N \in \mathbb{N}$,

$$\mathbb{E}[(X_N^*)^p]^{1/p} = \| X_N^* \|_{L^p} \leq p' \, \|X_N\|_{L^p} = \frac{p}{p-1} \mathbb{E}[(X_N)^p]^{1/p}.$$

Cor: (Doob's Martingale Inequality)
1. If $(M_n)_{n \geq 0}$ is a martingale, $\mathbb{P}(|M|_N^* \geq a) \leq \frac{1}{a}\mathbb{E}[|M_N| : M_N^* \geq a] \leq \frac{1}{a}\mathbb{E}[|M_N|]$
2. and if $M_n \in L^p$ for some $p > 1$ then $\| |M|_N^* \|_{L^p} \leq p' \| M_N \|_{L^p}$.

Pf. 1. Apply Lévy
2. Apply Doob
to the submartingale $X_n = |M_n|$.

Lemma: Let X, Y be ≥ 0 rv's with $\mathbb{P}(Y \geq y) \leq \frac{1}{y} \mathbb{E}[X : Y \geq y]$ $\forall y > 0$.

Let $p > 1$. Assume $Y \in L^p$. Then $\mathbb{E}[Y^p] \leq (p')^p \, \mathbb{E}[X^p]$.

Pf. $Y^p - 0^p = \int_0^Y p y^{p-1} \, dy = p \int_0^\infty \mathbb{1}\{y \leq Y\} \, y^{p-1} \, dy$

By Tonelli's theorem, $\mathbb{E}[Y^p] = p \int_0^\infty \mathbb{E}[\mathbb{1}_{\{y \leq Y\}}] \, y^{p-1} \, dy = p \int_0^\infty \mathbb{P}(Y \geq y) \, y^{p-1} \, dy$

$\leq p \int_0^\infty \mathbb{E}[X : Y \geq y] \, y^{p-2} \, dy$

$$\mathbb{E}[Y^p] \leq p \, \mathbb{E}\left[\int_0^\infty X \mathbb{1}_{Y \geq y} \, y^{p-2} \, dy\right]$$

$$= p \, \mathbb{E}\left[X \int_0^Y y^{p-2} \, dy\right] = \frac{p}{p-1} \mathbb{E}[X Y^{p-1}].$$

p'

Now we use Hölder's inequality.

$$\mathbb{E}[X Y^{p-1}] \leq \|X\|_{L^p} \, \|Y^{p-1}\|_{L^{p'}}$$

$$\|Y^{p-1}\|_{L^{p'}} = \mathbb{E}[(Y^{p-1})^{p'}]^{1/p'}$$

$$\leq \|X\|_{L^p} \left(\|Y\|_{L^p}^p\right)^{1/p'} = \|X\|_{L^p} \, \|Y\|_{L^p}^{p-1}$$

So $\|Y\|_{L^p}^{\cancel{p}\,1} \leq p' \, \|X\|_{L^p} \, \|Y\|_{L^p}^{\cancel{p-1}}$.

///

Theorem: (Doob's Submartingale Inequality)

Let $(X_n)_{n\geq 0}$ be a ≥ 0 submartingale, and let $p>1$. Then $\forall N \in \mathbb{N}$,

$$\mathbb{E}[(X_N^*)^p] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}[(X_N)^p] \quad \leftarrow \text{if } X_N \notin L^p \text{, no content.}$$

Pf. Since $X_n = |X_n|$ is a submartingale, by Lévy's maximal inequality,

$$\mathbb{P}(X_N^* \geq a) \leq \frac{1}{a}\mathbb{E}[X_N ; X_N^* \geq a]$$

$\uparrow X$ $\quad \uparrow Y$ $\quad \geq 0$

$Y = \max\{X_0, \dots, X_N\}$

$\leq X_0 + \dots + X_N \in L^p$.

$\therefore$ By Lemma, $\mathbb{E}[Y^p] \leq (p')^p \mathbb{E}[X^p]$. ///.

Sidenote: there is no L^1 (sub)martingale inequality.

E.g. S_n = symmetric random walk starting at 1.

$X_n = S_n^{\tau_0} \leftarrow$ martingale (by Optional Stopping), ≥ 0.

$$\mathbb{E}[X_n] = \mathbb{E}[S_n^{\tau_0}] = \mathbb{E}[S_{\tau_0 \wedge n}] = \mathbb{E}[S_0] = 1.$$

What can we say about $\mathbb{E}[X_n^*]$?

We proved [Lec 43.2] $\mathbb{P}^1(\tau_a < \tau_0) = \frac{1}{a-0} \quad \forall a > 0$.

$\mathbb{P}(\max_{m \geq 0} X_m \geq a)$

$$\mathbb{E}[\max_{m\geq 0} X_m] = \sum_{a=1}^{\infty} \mathbb{P}(\max_{m \geq 0} X_m \geq a) \quad \text{(See [Lec 46.3])}$$

$$= \infty.$$

But $X_n^* = \max\{X_0, \dots, X_n\} \uparrow \max_{m \geq 0} X_m$, so by MCT, $\mathbb{E}[X_n^*] \uparrow \infty$

while $\mathbb{E}[X_n] = 1 \quad \forall n$.

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