Another basic integration theory tool we haven't needed (until now) is **Hölder's Inequality**.

Def: Given $1 < p < \infty$, the **conjugate exponent** p' is defined by

$$\frac{1}{p} + \frac{1}{p'} = 1$$

By convention, we extend to $1 \leq p \leq \infty$, with $1' = \infty$, $\infty' = 1$.
Note: $p'' = p$.

Theorem: (Hölder's Inequality)
Let $1 \leq p \leq \infty$. If f, g are measurable wrt $(\Omega, \mathcal{F}, \mu)$ then

$$\|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^{p'}}.$$

Here $\|f\|_{L^\infty} := \operatorname{ess\,sup} |f|$
So if p (or p') $= 1$, this is just

$$\int |fg| \, d\mu$$

If $p = 2$, this is the Cauchy-Schwarz inequality.

The proof requires one elementary convexity result.

Lemma: If $s,t \geq 0$, $1<p<\infty$, then $st \leq \frac{1}{p}s^p + \frac{1}{p'}t^{p'}$

Pf. exp is a convex function. $\therefore$ since $\frac{1}{p}+\frac{1}{p'}=1$,

$$st = e^{\ln s}e^{\ln t} = e^{\ln s + \ln t}$$

Proof of Hölder's Inequality $\|fg\|_1 \leq \|f\|_p \|g\|_{p'}$

- Already covered the case $p=1,\infty$.
- If $\|f\|_p=0$ or $\|g\|_{p'}=0$, $fg=0$ a.s. and so Hölder reads $0\leq 0$
- Assume $1<p<\infty$ and $0<\|f\|_p, \|g\|_{p'}<\infty$

$$s := \frac{|f|}{\|f\|_p} \qquad t := \frac{|g|}{\|g\|_{p'}}$$

$$\frac{|fg|}{\|f\|_p\|g\|_{p'}} = st \leq \frac{1}{p}s^p + \frac{1}{p'}t^{p'} = \frac{1}{p}\frac{|f|^p}{\|f\|_p^p} + \frac{1}{p'}\frac{|g|^{p'}}{\|g\|_{p'}^{p'}}$$