Another basic integration theory tool we haven't needed (until now) is **Hölder's Inequality**.

Def: Given $1<p<\infty$, the **conjugate exponent** p' is defined by

$$\frac{1}{p}+\frac{1}{p'}=1 \qquad p'=\frac{p}{p-1}$$

By convention, we extend to $1\leq p\leq\infty$, with $1'=\infty$, $\infty'=1$.

Note: $p''=p$.

Theorem: (Hölder's Inequality)

Let $1\leq p\leq\infty$. If f,g are measurable wrt $(\Omega,\mathcal{F},\mu)$ then

if $|fg|\in L^1$ $\quad \left|\int fg\,d\mu\right| \leq \|fg\|_{L^1} \leq \|f\|_{L^p}\|g\|_{L^{p'}}$.

Here $\|f\|_{L^\infty} := \operatorname{ess\,sup}|f| = \inf\{a\geq 0 : \mu\{|f|>a\}=0\}$

So if p (or p') $=1$, this is just

$$\int|fg|\,d\mu \leq \int|f|\operatorname{ess\,sup}|g|\,d\mu = \|g\|_{L^\infty}\|f\|_{L^1}.$$

If $p=p'=2$, this is the Cauchy-Schwarz inequality.

The proof requires one elementary convexity result.

Lemma: If $s,t \geq 0$, $1<p<\infty$, then $st \leq \frac{1}{p}s^p + \frac{1}{p'}t^{p'}$.

Pf. exp is a convex function. $\therefore$ since $\frac{1}{p}+\frac{1}{p'}=1$,

$$st = e^{\ln s}e^{\ln t} = e^{\ln s + \ln t} = e^{\frac{1}{p}\ln(s^p) + \frac{1}{p'}\ln(t^{p'})} \leq \frac{1}{p}e^{\ln(s^p)} + \frac{1}{p'}e^{\ln(t^{p'})}$$

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Proof of Hölder's Inequality $\|fg\|_1 \leq \|f\|_p \|g\|_{p'}$

- Already covered the case $p=1,\infty$.
- If $\|f\|_p = 0$ or $\|g\|_{p'}=0$, $fg=0$ a.s. and so Hölder reads $0\leq 0$
- Assume $1<p<\infty$ and $0<\|f\|_p, \|g\|_{p'}<\infty$

$$s := \frac{|f|}{\|f\|_p} \qquad t := \frac{|g|}{\|g\|_{p'}}$$

$$\int \frac{|fg|}{\|f\|_p\|g\|_{p'}} = st \leq \frac{1}{p}s^p + \frac{1}{p'}t^{p'} = \int\left(\frac{1}{p}\frac{|f|^p}{\|f\|_p^p} + \frac{1}{p'}\frac{|g|^{p'}}{\|g\|_{p'}^{p'}}\right)d\mu$$

$$\frac{\int |fg| d\mu}{\|f\|_p\|g\|_{p'}} \leq \frac{1}{p}\frac{\int |f|^p d\mu}{\|f\|_p^p} + \frac{1}{p'}\frac{\int |g|^{p'}d\mu}{\|g\|_{p'}^{p'}} = \frac{1}{p}+\frac{1}{p'} = 1.$$

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