# Simple stock market model:

A stock evolves in time, north Xn per share at time n. You (investor) buy U shares of the stock at time s, and sell them at time t>s. How much profit/loss did you incur?

### $U \cdot (X_t - X_5)$

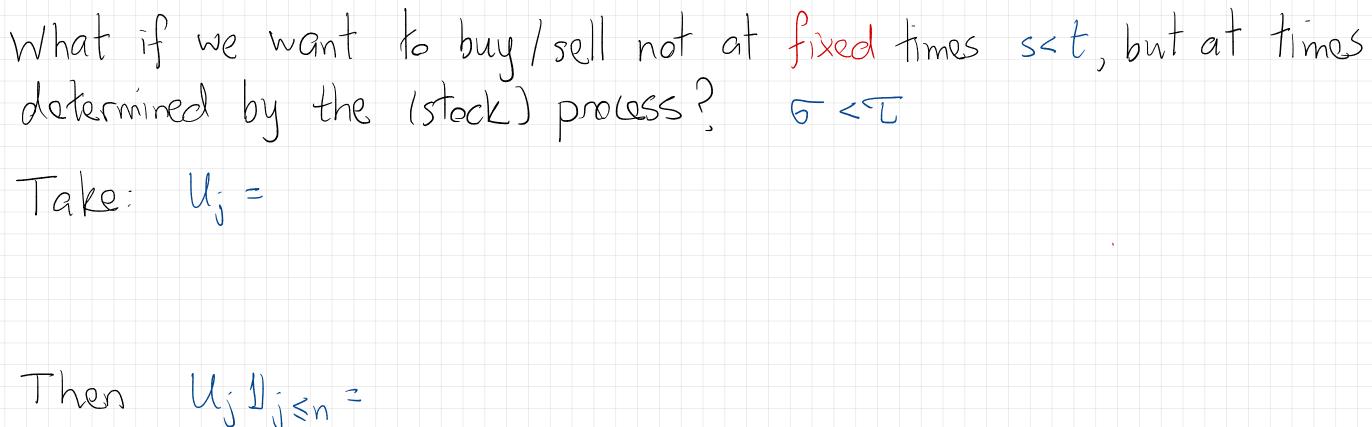
We could accomplish the same transaction buying and selling U shares at each time step. Or we can vary the amount we by/sell per step.

Def. Let (Un) , and (Xn) be R-segnences. For n 21,

 $= \sum_{j=1}^{n} (U, X) := \sum_{j=1}^{n} U_j (X_j - X_{j-1}) = \sum_{j=1}^{n} U_j \Delta X_j$ 

Your profit /loss (buying U; shares at time j-1 and selling them at time j) up to time n.

(Discrete) "Stochastic Integral"



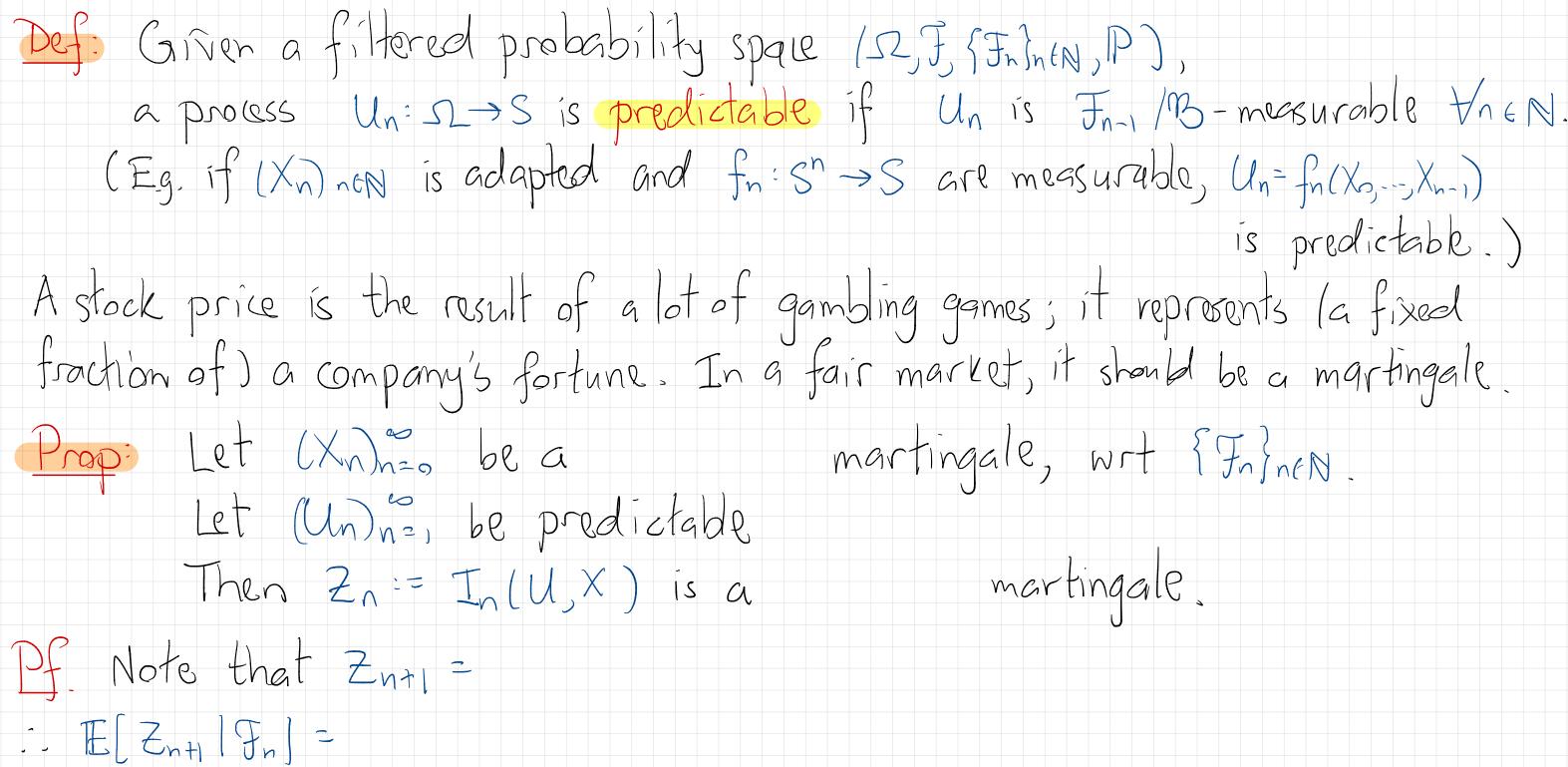
So  $I_n(u, x)$ 

:- (at least formally), lim In(U,X) =

If (Xn)nzo and (Un)nz, are stochastic processes,

what kind of process is Zn=In(U,X)?

Key point: Let  $\mathcal{F}_n = \sigma(X_q, X_{q_1}, \dots, X_n)$ . As  $U_j$  is the # shares bought at time j-1, we must have  $U_j$   $\mathcal{F}_{j-1}$ -measurable.



is predictabe.)



 $(\mathbf{A}) \in [\mathbf{I}_n(\mathbf{u}, \mathbf{X})] = 0$  Then  $(\mathbf{X}_n)_{n \ge 0}$  is a martingale

Pf As in the previous proof,  $E[I_{n+1}(\mathcal{U}, X)]F_{n}] = I_{n}(\mathcal{U}, X) + U_{n+1}E[X_{n+1}-X_{n}]F_{n}]$ . Take expectations: · In the = 0 case of (A)

take Unti =

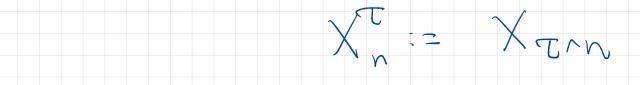
take U;=

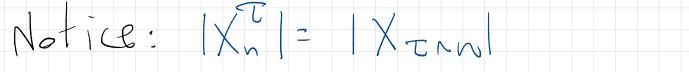
. In the 2/60 case of (A), fix n and BEFn, and

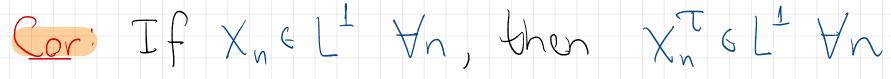


# Let (Xn)nzo be an adapted process, and let TE NUEcoz be a stopping time.

The stopped process (XT) n=0 is defined by





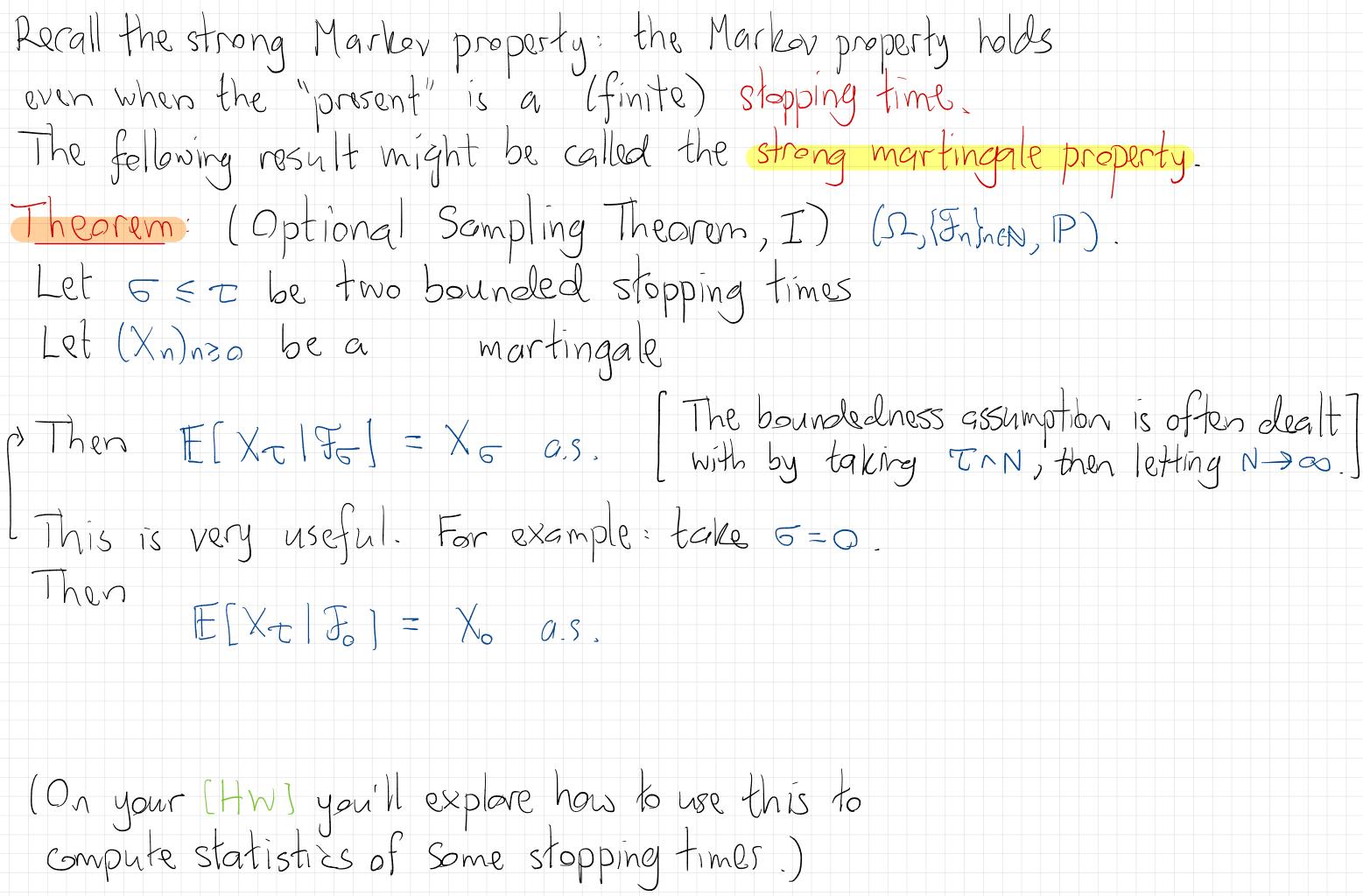


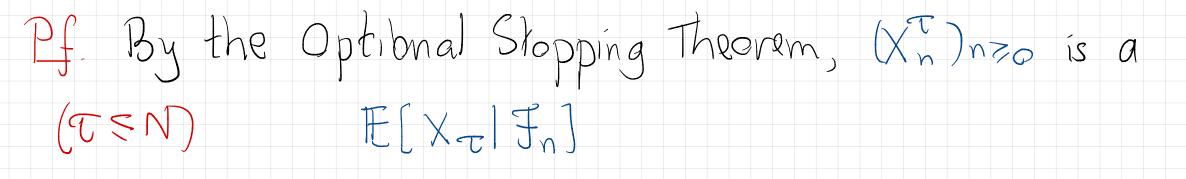
Theorem (Optional Stopping Theorem)

Let (Xn)nzo be a martingale

Let T be a stopping time. Then (XT) nzo is glso a martinggle.

Pf. Un= Inst  $\therefore Z_n = I_n(U,X)$ 





# Now, recall how to condition E[ IFG] frem [Lec 45.3]:

## If YELT, and Yn = E[YIFn], then E[YIF6]=

 $: \mathbb{E}[X_{\tau}|\mathcal{F}_{c}] =$ 

## The boundedness condition cannot be dropped

Eg. (Xn)nzo = symmetric random walk. Let x + y + Z Then  $E^{\chi}[X_{\tau_y}] = E^{\chi}[X_o]$ 

we'll later see under what conditions optional Sampling holds for unbounded stopping times.

