Simple stock market model:

A stock evolves in time, worth X_n per share at time n.
You (investor) buy u shares of the stock at time s, and sell them at time $t>s$. How much profit / loss did you incur?

$$u\cdot(X_t - X_s)$$

We could accomplish the same transaction buying and selling u shares at each time step. Or we can vary the amount we buy/sell per step.

Def: Let $(U_n)_{n=1}^{\infty}$ and $(X_n)_{n=0}^{\infty}$ be $\mathbb{R}$-sequences. For $n\geq 1$,

$$I_n(U,X) := \sum_{j=1}^{n} U_j\,(X_j - X_{j-1}) = \sum_{j=1}^{n} U_j\,\Delta X_j$$

Your profit / loss (buying U_j shares at time $j-1$ and selling them at time j) up to time n.

(Discrete) "Stochastic Integral"

What if we want to buy / sell not at fixed times $s < t$, but at times determined by the (stock) process? $\sigma < \tau$

Take: $U_j =$

Then $U_j \mathbb{1}_{j \leq n} =$

So $I_n(U, X)$

$\therefore$ (at least formally), $\lim_{n \to \infty} I_n(U, X) =$

If $(X_n)_{n \geq 0}$ and $(U_n)_{n \geq 1}$ are stochastic processes, what kind of process is $Z_n = I_n(U, X)$?

Key point: Let $\mathcal{F}_n = \sigma(X_0, X_1, \ldots, X_n)$.
As U_j is the # shares bought at time $j-1$, we must have U_j $\mathcal{F}_{j-1}$-measurable.

Def: Given a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n\in\mathbb{N}}, \mathbb{P})$, a process $U_n : \Omega \to S$ is **predictable** if U_n is $\mathcal{F}_{n-1}/\mathcal{B}$-measurable $\forall n \in \mathbb{N}$.
(E.g. if $(X_n)_{n\in\mathbb{N}}$ is adapted and $f_n : S^n \to S$ are measurable, $U_n = f_n(X_0, \dots, X_{n-1})$ is predictable.)

A stock price is the result of a lot of gambling games; it represents (a fixed fraction of) a company's fortune. In a fair market, it should be a martingale.

Prop: Let $(X_n)_{n=0}^{\infty}$ be a martingale, wrt $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$.
Let $(U_n)_{n=1}^{\infty}$ be predictable.
Then $Z_n := I_n(U, X)$ is a martingale.

Pf. Note that $Z_{n+1} =$

$\therefore\ \mathbb{E}[Z_{n+1} \mid \mathcal{F}_n] =$

Prop: Suppose $(X_n)_{n\geq 0}$ is adapted and L^1. Moreover, suppose that for all bounded predictable processes $(U_n)_{n=1}^{\infty}$,

$(\star)$ $\mathbb{E}[I_n(U,X)] = 0$. Then $(X_n)_{n\geq 0}$ is a martingale.

Pf. As in the previous proof, $\mathbb{E}[I_{n+1}(U,X) \mid \mathcal{F}_n] = I_n(U,X) + U_{n+1}\mathbb{E}[X_{n+1} - X_n \mid \mathcal{F}_n]$.

Take expectations:

- In the $=0$ case of $(\star)$

  take $U_{n+1} :=$

- In the $\geq / \leq 0$ case of $(\star)$, fix n and $B \in \mathcal{F}_n$, and take $U_j =$

Stopped Processes

Let $(X_n)_{n\geq 0}$ be an adapted process, and let $\tau \in \mathbb{N}\cup\{\infty\}$ be a stopping time.
The stopped process $(X_n^\tau)_{n\geq 0}$ is defined by

$$X_n^\tau := X_{\tau\wedge n}$$

Notice: $|X_n^\tau| = |X_{\tau\wedge n}|$

Cor: If $X_n \in L^1$ $\forall n$, then $X_n^\tau \in L^1$ $\forall n$.

Theorem: (Optional Stopping Theorem)
Let $(X_n)_{n\geq 0}$ be a martingale.
Let τ be a stopping time. Then $(X_n^\tau)_{n\geq 0}$ is also a martingale.

Pf. $U_n = \mathbb{1}_{n\leq\tau}$

$\therefore$ $Z_n = I_n(U,X)$

Recall the strong Markov property: the Markov property holds even when the "present" is a (finite) stopping time.

The following result might be called the strong martingale property.

Theorem: (Optional Sampling Theorem, I) $(\Omega, \{\mathcal{F}_n\}_{n\in\mathbb{N}}, \mathbb{P})$.

Let $\sigma \le \tau$ be two bounded stopping times

Let $(X_n)_{n\ge 0}$ be a martingale

Then $\mathbb{E}[X_\tau \mid \mathcal{F}_\sigma] = X_\sigma$ a.s.

[The boundedness assumption is often dealt with by taking $\tau \wedge N$, then letting $N \to \infty$.]

This is very useful. For example: take $\sigma = 0$.

Then

$$\mathbb{E}[X_\tau \mid \mathcal{F}_0] = X_0 \quad a.s.$$

(On your [HW] you'll explore how to use this to compute statistics of some stopping times.)

Pf. By the Optional Stopping Theorem, $(X^{\tau}_n)_{n\geq 0}$ is a martingale; so

$(\tau \leq N)$ $\mathbb{E}[X_\tau | \mathcal{F}_n]$

Now, recall how to condition $\mathbb{E}[\cdot | \mathcal{F}_\sigma]$ from [Lec 45.3]:

If $Y \in L^1$, and $Y_n := \mathbb{E}[Y | \mathcal{F}_n]$, then $\mathbb{E}[Y | \mathcal{F}_\sigma] =$

$\therefore \mathbb{E}[X_\tau | \mathcal{F}_\sigma] =$

The boundedness condition cannot be dropped

Eg. $(X_n)_{n\geq 0}$ = symmetric random walk. Let $x \neq y \in \mathbb{Z}$.

Then $\mathbb{E}^x[X_{\tau_y}]$ $\mathbb{E}^x[X_0]$

We'll later see under what conditions Optional Sampling holds for unbounded stopping times.