Simple stock market model:

A stock evolves in time, worth X_n per share at time n.
You (investor) buy U shares of the stock at time s, and sell them at time $t > s$. How much profit / loss did you incur?

$$U \cdot (X_t - X_s) = U \cdot ((X_t - X_{t-1}) + (X_{t-1} - X_{t-2}) + \cdots + (X_{s+1} - X_s))$$

We could accomplish the same transaction buying and selling U shares at each time step. Or we can vary the amount we buy/sell per step.

Def: Let $(U_n)_{n=1}^{\infty}$ and $(X_n)_{n=0}^{\infty}$ be $\mathbb{R}$-sequences. For $n \geq 1$,

$$I_n(U, X) := \sum_{j=1}^{n} U_j (X_j - X_{j-1}) = \sum_{j=1}^{n} U_j \Delta X_j$$

Your profit / loss (buying U_j shares at time $j-1$ and selling them at time j) up to time n.

(Discrete) "Stochastic Integral"

What if we want to buy / sell not at fixed times $s<t$, but at times determined by the (stock) process? $\sigma < \tau$ E.g. σ = 1st time $X \leq \$5$
τ = 1st time $X \geq \$10$.

Take: $U_j = \mathbb{1}_{\sigma < j \leq \tau} = \mathbb{1}_{(\sigma,\tau]}(j)$

"stochastic interval" $\{(\omega,j) : \sigma(\omega) < j \leq \tau(\omega)\}$ σ, τ stopping times.

Then $U_j \mathbb{1}_{j\leq n} = \mathbb{1}_{(\sigma\wedge n,\, \tau\wedge n]}(j)$.

So $I_n(U,X) = \sum_{j=1}^{n} \mathbb{1}_{\sigma\wedge n < j \leq \tau \wedge n} (X_j - X_{j-1}) = X_{\tau\wedge n} - X_{\sigma \wedge n}$.

$\therefore$ (at least formally), $\lim_{n\to\infty} I_n(U,X) = X_\tau - X_\sigma \geq \$5$.

If $(X_n)_{n\geq 0}$ and $(U_n)_{n\geq 1}$ are stochastic processes, what kind of process is $Z_n = I_n(U,X)$?

Key point: Let $\mathcal{F}_n = \sigma(X_0, X_1, \dots, X_n)$.
As U_j is the # shares bought at time $j-1$, we must have U_j $\mathcal{F}_{j-1}$-measurable.

Def: Given a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n\in\mathbb{N}}, \mathbb{P})$, a process $U_n : \Omega \to S$ is **predictable** if U_n is $\mathcal{F}_{n-1}/\mathcal{B}$-measurable $\forall n \in \mathbb{N}$.

(E.g. if $(X_n)_{n\in\mathbb{N}}$ is adapted and $f_n : S^n \to S$ are measurable, $U_n = f_n(X_0, \dots, X_{n-1})$ is predictable.)

A stock price is the result of a lot of gambling games; it represents (a fixed fraction of) a company's fortune. In a fair market, it should be a martingale.

Prop: Let $(X_n)_{n=0}^{\infty}$ be a (sub/super) martingale, wrt $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$.
Let $(U_n)_{n=1}^{\infty}$ be predictable (and ≥ 0).
Then $Z_n := I_n(U, X)$ is a (sub/super) martingale.

Pf. Note that $Z_{n+1} = Z_n + U_{n+1}(X_{n+1} - X_n)$

$$\therefore \quad \mathbb{E}[Z_{n+1} \mid \mathcal{F}_n] = \underbrace{\mathbb{E}[Z_n \mid \mathcal{F}_n]}_{Z_n} + \underbrace{\mathbb{E}[U_{n+1}(X_{n+1} - X_n) \mid \mathcal{F}_n]}_{U_{n+1}\,\mathbb{E}[X_{n+1} - X_n \mid \mathcal{F}_n]}$$

$$\underset{\mathcal{F}_n}{Z_n} = \sum_{j=1}^{n} \underset{\mathcal{F}_{j-1}}{U_j}\,(\underset{\mathcal{F}_j}{X_j} - \underset{\mathcal{F}_{j-1}}{X_{j-1}})$$

$$Z_n + U_{n+1}\underbrace{\mathbb{E}[X_{n+1} - X_n \mid \mathcal{F}_n]}_{\substack{= 0 \quad \text{mart} \\ \geq 0 \quad \text{sub} \\ \leq 0 \quad \text{super}}}$$

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Prop: Suppose $(X_n)_{n\geq 0}$ is adapted and L^1. Moreover, suppose that for all bounded $(\geq)$ predictable processes $(U_n)_{n=1}^{\infty}$,

$$(\star)\quad \mathbb{E}[I_n(U,X)] \overset{\geq}{\underset{\leq}{=}} 0.$$

Then $(X_n)_{n\geq 0}$ is a $\overset{\text{sub}}{\underset{\text{super}}{}}$ martingale.

Pf. As in the previous proof, $\mathbb{E}[I_{n+1}(U,X)\mid\mathcal{F}_n] = I_n(U,X) + U_{n+1}\mathbb{E}[X_{n+1}-X_n\mid\mathcal{F}_n]$.

Take expectations: $\underbrace{\mathbb{E}[I_{n+1}(U,X)]}_{=0} = \underbrace{\mathbb{E}[I_n(U,X)]}_{=0} + \mathbb{E}[U_{n+1}\mathbb{E}[X_{n+1}-X_n\mid\mathcal{F}_n]]$

- In the $=0$ case of $(\star)$

  take $U_{n+1} := \operatorname{sgn}(\mathbb{E}[X_{n+1}-X_n\mid\mathcal{F}_n]) \leftarrow \mathcal{F}_n$-meas, $\therefore (U_n)_{n\geq 0}$ predictable.

  $\therefore \|\mathbb{E}[X_{n+1}-X_n\mid\mathcal{F}_n]\|_{L^1} = \mathbb{E}(|\mathbb{E}[X_{n+1}-X_n\mid\mathcal{F}_n]|) = 0$

  $\Rightarrow (X_n)_{n\geq 0}$ is a martingale.

- In the $\geq/\leq 0$ case of $(\star)$, fix n and $B\in\mathcal{F}_n$, and take $U_j = \delta_{n+1,j}\mathbb{1}_B$ predictable. $I_{n+1}(U,X) = (X_{n+1}-X_n)\mathbb{1}_B$.

  $\rightarrow \therefore \mathbb{E}[X_{n+1}-X_n : B] = \mathbb{E}[I_{n+1}(U,\cancel{X})] \overset{\geq 0}{\underset{\leq 0}{}}$

  True $\forall B\in\mathcal{F}_n$: $\therefore \mathbb{E}[X_{n+1}-X_n\mid\mathcal{F}_n] \overset{\geq 0 \therefore \text{sub}}{\underset{\leq 0 \therefore \text{super}}{}}$. ///

Stopped Processes

Let $(X_n)_{n\geq 0}$ be an adapted process, and let $\tau \in \mathbb{N} \cup \{\infty\}$ be a stopping time.
The stopped process $(X_n^\tau)_{n\geq 0}$ is defined by

$$X_0 + Z_n = X_n^\tau := X_{\tau\wedge n} \leftarrow \leq n$$

Notice: $|X_n^\tau| = |X_{\tau\wedge n}| = \left| \sum_{k=0}^{n} \mathbb{1}_{\{\tau=k\}} X_k \right| \leq \sum_{k=0}^{n} \mathbb{1}_{\{\tau=k\}} |X_k| \leq \sum_{k=0}^{n} |X_k|$.

Cor: If $X_n \in L^1 \ \forall n$, then $X_n^\tau \in L^1 \ \forall n$.

Theorem: (Optional Stopping Theorem)
Let $(X_n)_{n\geq 0}$ be a (sub/super) martingale.
Let τ be a stopping time. Then $(X_n^\tau)_{n\geq 0}$ is also a (sub/super) martingale.

Pf. $U_n = \mathbb{1}_{n\leq\tau} = 1 - \mathbb{1}_{\tau<n} = 1 - \mathbb{1}_{\tau\leq n-1}$ is predictable.
$\therefore$ $Z_n = I_n(U,X)_n$ is a (sub/super) martingale.

$$\sum_{j=1}^{n} U_j (X_j - X_{j-1}) = \sum_{j=1}^{n} \mathbb{1}_{j\leq\tau} (X_j - X_{j-1}) = \sum_{j=1}^{\tau\wedge n} (X_j - X_{j-1}) = X_{\tau\wedge n} - X_0.$$

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Recall the strong Markov property: the Markov property holds even when the "present" is a (finite) stopping time.

The following result might be called the strong martingale property.

Theorem: (Optional Sampling Theorem, I) $(\Omega, \{\mathcal{F}_n\}_{n\in\mathbb{N}}, \mathbb{P})$.

Let $\sigma \le \tau$ be two bounded stopping times ($\exists\, N<\infty$ s.t. $\tau \le N$ a.s.)

Let $(X_n)_{n\ge 0}$ be a $\substack{\text{sub}\\ \\ \text{super}}$ martingale

Then $\mathbb{E}[X_\tau \mid \mathcal{F}_\sigma] \;\substack{\le \\ = \\ \ge}\; X_\sigma$ a.s.

[The boundedness assumption is often dealt with by taking $\tau \wedge N$, then letting $N \to \infty$.]

This is very useful. For example: take $\sigma = 0$.

Then

$$\mathbb{E}\big[\,\mathbb{E}[X_\tau \mid \mathcal{F}_0]\,\big] = \mathbb{E}[\,X_0\,] \quad \text{a.s.}$$

$$\mathbb{E}[X_\tau] = \mathbb{E}[X_0].$$

(On your [HW] you'll explore how to use this to compute statistics of some stopping times.)

Pf. By the Optional Stopping Theorem, $(X^\tau_n)_{n\geq 0}$ is a $\substack{\text{sub}\\\text{super}}$ martingale; so

$(\tau \leq N)$ $$\mathbb{E}[X_\tau | \mathcal{F}_n] = \mathbb{E}[X_{\tau\wedge N} | \mathcal{F}_n] = \mathbb{E}[X^\tau_N | \mathcal{F}_n]$$

$$\underset{\geq}{\overset{\leq}{=}} X^\tau_{n\wedge N}. \quad \forall n \leq \infty.$$

Now, recall how to condition $\mathbb{E}[\,\cdot\,|\mathcal{F}_\sigma]$ from [Lec 45.3]:

If $Y \in L^1$, and $Y_n := \mathbb{E}[Y|\mathcal{F}_n]$, then $\mathbb{E}[Y|\mathcal{F}_\sigma] = Y_\sigma = \sum_{n\leq\infty} \mathbb{1}_{\{\sigma=n\}} \mathbb{E}[Y|\mathcal{F}_n]$.

$$\therefore \mathbb{E}[X_\tau | \mathcal{F}_\sigma] = \sum_{n\leq\infty} \mathbb{1}_{\{\sigma=n\}} \mathbb{E}[X_\tau|\mathcal{F}_n]$$

$\sigma \leq \tau \leq N$

$$\underset{\geq}{\overset{\leq}{=}} \sum_{n\leq\infty} \mathbb{1}_{\{\sigma=n\}} X^\tau_{n\wedge N} = X^\tau_{\sigma\wedge N} = X_{\tau\wedge(\sigma\wedge N)} = X_\sigma. \quad ///$$

The boundedness condition cannot be dropped

Eg. $(X_n)_{n\geq 0}$ = symmetric random walk. Let $x \neq y \in \mathbb{Z}$.

Then
$$y = \mathbb{E}^x[X_{\tau_y}] \neq \mathbb{E}^x[X_0] = x.$$

We'll later see under what conditions Optional Sampling holds for unbounded stopping times.