There are several ways to produce (sub/super) martingales from other martingales (or other processes).

Prop: Let $(X_n)_{n\in\mathbb{N}}$ be a (sub)martingale, and let $\varphi:\mathbb{R}\to\mathbb{R}$ be convex & increasing s.t. $\varphi(X_n)\in L^1$ $\forall n$. Then $(\varphi(X_n))_{n\in\mathbb{N}}$ is a submartingale.

Pf. $\varphi(X_n) = \varphi\left(E_{\mathcal{F}_n}[X_{n+1}]\right) \leq E_{\mathcal{F}_n}[\varphi(X_{n+1})]$

If we only know $(X_n)_{n\in\mathbb{N}}$ is a submartingale, $X_n \leq E_{\mathcal{F}_n}[X_{n+1}]$

$\varphi\uparrow$, $\therefore \varphi(X_n) \leq \varphi\left(E_{\mathcal{F}_n}[X_{n+1}]\right)$

$\leq E_{\mathcal{F}_n}[\varphi(X_{n+1})]$. ///

Eg. If $(X_n)_{n\in\mathbb{N}}$ is an L^p martingale $(p\geq 1)$ then $(|X_n|^p)_{n\in\mathbb{N}}$ is a submartingale.

If $X_n\geq 0$ is an L^p submartingale, $(X_n^p)_{n\geq 0}$ is.

We've seen a number of Markov chains that are also martingales. That's not general: even if a Markov chain's state space is $\subseteq\mathbb{R}$, it's usually not a martingale.

<u>But</u> some functions of it are.

Theorem: Let $(X_n)_{n\in\mathbb{N}}$ be a time-homogeneous Markov chain in $(S,\mathcal{B})$ with transition operator Q. Let $f: \mathbb{N}\times S \to \mathbb{R}$ be measurable, satisfying

$$\mathbb{E}[\,|f(n,X_n)|\,] < \infty \quad \forall n.$$

Then $Z_n := f(n,X_n)$ is a (sub) martingale, provided

$$(Qf)(n+1,\cdot) \overset{(\leq)}{=} f(n,\cdot) \quad \forall n\in\mathbb{N}$$

(In particular, if $f: S\to\mathbb{R}$ doesn't depend on n, the condition is $Qf = f$

Pf. $\mathbb{E}[Z_{n+1}|\mathcal{F}_n] = \mathbb{E}[\,f(n+1,X_{n+1})\,|\mathcal{F}_n] = Qf(n+1,X_n) \overset{(\leq)}{=} f(n,X_n) = Z_n$. ///

One way to make this work, at least for a finite time interval $0\leq n\leq T$, is to run the Markov chain backwards

$$f(n,y) := (Q^{T-n}g)(y) \quad \text{for some } g: S\to\mathbb{R}.$$

$$Qf(n+1,\cdot) = Q\left(Q^{T-(n+1)}g\right) = Q^{T-n}g = f(n,\cdot)$$

So $\left(Z_n = (Q^{T-n}g)(X_n)\right)_{0\leq n\leq T}$ is a martingale.

Eg. $(X_n)_{n\in\mathbb{N}} = RW(p)$ on $\mathbb{Z}$, $Qf(x) = pf(x+1) + (1-p)f(x-1)$, with $X_0 = 0$

We've seen that all (and only) the functions $f(x) = \alpha + \beta\left(\frac{1-p}{p}\right)^x$ satisfy $Qf = f$ on $\mathbb{Z}$.

$$\therefore\ M_n = \alpha + \beta\left(\frac{1-p}{p}\right)^{X_n} \text{ is a martingale } \forall \alpha, \beta \in \mathbb{R}.$$

We can also verify this directly. Set $\lambda = \frac{1-p}{p}$. $X_n = \xi_1 + \cdots + \xi_n$ iid $\xi_n \overset{d}{=} p\delta_1 + (1-p)\delta_{-1}$

$$\mathbb{E}_{\mathcal{F}_n}[\lambda^{X_{n+1}}] = \mathbb{E}_{\mathcal{F}_n}[\lambda^{X_n + \xi_{n+1}}] = \lambda^{X_n}\mathbb{E}_{\cancel{\mathcal{F}_n}}[\lambda^{\xi_{n+1}}] = \lambda^{X_n}(p\lambda^1 + (1-p)\lambda^{-1})$$

$$\cancel{p}\frac{1-p}{\cancel{p}} + (1-p)\frac{p}{1-p} = 1$$

(Note: {martingales} is a vector space.)

Now, can we get more leverage if we allow f to depend on n?

Notice: if $\lambda \neq 0$, $Q(\lambda^x) = p\lambda^{x+1} + (1-p)\lambda^{x-1}$

$$= (p\lambda + (1-p)\lambda^{-1})\lambda^x$$

So "Q^{-1}" makes sense (on the span of exponential functions)

and we can define $f_\lambda(n,x) := (p\lambda + (1-p)\lambda^{-1})^{-n}\lambda^x$

$$= Q^{-n}\lambda^x$$

Thus $Qf_\lambda(n+1, \cdot) = f_\lambda(n, \cdot)$ $\forall n$. So, if X_0 is bounded,

$$M_n = f_\lambda(n, X_n) = (p\lambda + (1-p)\lambda^{-1})^{-n}\lambda^{X_n} \text{ is a martingale.}$$

$f_{e^\theta}(n,x) = (pe^\theta + (1-p)e^{-\theta})^{-n} e^{\theta x}$ satisfies $Q f_{e^\theta}(n+1,\cdot) = f_{e^\theta}(n,\cdot)$, $\forall \lambda \neq 0$.

depends smoothly on θ. $Qf(x) = p\,f(x+1) + (1-p) f(x-1)$

action commutes with derivatives.

$\therefore$ $\frac{d^k}{d\theta^k} f_{e^\theta}(n,x)\big|_{\theta=0}$ also satisfies the recurrence.

A careful computation, and argument, then shows

Prop: If $(X_n)_{n\in\mathbb{N}}$ is $RW(p)$ (Markov chain; not a martingale if $p \neq \frac{1}{2}$)

then

$$M_n^{(1)} = X_n - n(p-(1-p))$$

$$M_n^{(2)} = (M_n^{(1)})^2 - 4np(1-p)$$

are martingales. [HW]

We will soon see: this provides some very effective tools to calculate expectations (and higher moments) of some stopping times.