Uniform Integrability

A collection of random variables $\Lambda \subseteq L^1(\Omega, \mathcal{F}, \mathbb{P})$ is called uniformly integrable (UI) if

$$E[\,|X| : |X| \geq a\,]$$

I.e. tail expectations are uniformly small.

Eg. If Λ has a dominating L^1 function, $|X| \leq g \in L^1$ $\forall X \in \Lambda$,

Eg. If $\Lambda \subseteq L^p$ for some $p > 1$ and $\sup_{X \in \Lambda} \|X\|_{L^p} < \infty$,
then Λ is UI.

Eg. Any subset of a UI set is UI.

Lemma: If $\Lambda \subseteq L^1$ is UI, then Λ is L^1-bounded: $\sup_{X\in\Lambda} \mathbb{E}[|X|] < \infty$.

Pf. By assumption, $\sup_{X\in\Lambda} \mathbb{E}[|X| \mathbb{1}_{|X|\geq a}] \to 0$ as $a \to \infty$.

So fix a s.t.

The converse is false, as we'll soon see.

UI is equivalent to another uniform regularity condition.

Def: A collection of random variables $\Lambda \subseteq L^1(\Omega, \mathcal{F}, \mathbb{P})$ is called **uniformly absolutely continuous** (UAC) if

$\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\forall B \in \mathcal{F}$, $\mathbb{P}(B) < \delta \Rightarrow \sup_{X\in\Lambda} \mathbb{E}[|X| : B] < \varepsilon$.

i.e.

$$\limsup_{\delta\downarrow 0} \left\{ \mathbb{E}[|X| : B] \,;\, X\in\Lambda,\ \mathbb{P}(B) < \delta \right\} = 0.$$

Prop: For any $\Lambda \subseteq L^1(\Omega, \mathcal{F}, \mathbb{P})$, Λ is UI iff Λ is UAC and L^1-bounded.

Pf. ($\Longrightarrow$) For any $a > 0$, $B \in \mathcal{F}$, $X \in \Lambda$,

$$\mathbb{E}[|X| : B] =$$

$$\therefore \limsup_{\delta \downarrow 0} \{ \mathbb{E}[|X| : B] : X \in \Lambda, \mathbb{P}(B) < \delta \} \leq$$

We already showed that UI sets are L^1-bounded.

($\Longleftarrow$) Let $K = \sup_{X \in \Lambda} \|X\|_{L^1}$. For $a > 0$, $X \in \Lambda$,

$$\mathbb{P}(|X| \geq a) \leq \frac{\|X\|_{L^1}}{a} \leq \frac{K}{a}.$$

$\forall \varepsilon > 0$, choose $\delta > 0$ s.t. $\mathbb{P}(B) < \delta \Rightarrow \sup_{X \in \Lambda} \mathbb{E}[|X| : B] < \varepsilon$,

Now, set $a =$

Cor: If $\Lambda \subseteq L^1$ is UI and $X \in L^1$, then $\Lambda + X = \{Y+X : Y \in \Lambda\}$ is UI.

Pf. By the last proposition, Λ is UAC.

Fix $\varepsilon > 0$, and choose $\delta_1 > 0$ s.t. $\mathbb{P}(B) < \delta_1 \Rightarrow \mathbb{E}[|Y| : B] < \varepsilon/2 \ \forall Y \in \Lambda$.

Of course $\{X\}$ is UI, $\therefore$ UAC, so choose $\delta_2 > 0$ s.t. $\mathbb{P}(B) < \delta_2 \Rightarrow \mathbb{E}[|X| : B] < \frac{\varepsilon}{2}$.

$\therefore$ For $\delta = \delta_1 \wedge \delta_2$,

$$\mathbb{E}[|X+Y| : B]$$

$\therefore \Lambda + X$ is UAC.

Also $\sup\{\|X+Y\|_{L^1} : Y \in \Lambda\}$

$\therefore \Lambda + X$ is UI.

Uniform Integrability is precisely the gap between L^1-convergence and convergence in probability.

Theorem: (Vitali Convergence Theorem)

Let $\{X_n\}_{n=1}^{\infty} \subset L^1(\Omega, \mathcal{F}, \mathbb{P})$, and let X be measurable.

Then

$$X \in L^1 \text{ and } X_n \to X \text{ in } L^1$$

$$\text{iff}$$

$$\{X_n\}_{n=1}^{\infty} \text{ is UI and } X_n \xrightarrow{P} X$$

Pf. If $X_n \to X$ in L^1, then $X_n \to_{\mathbb{P}} X$ [Lec. 13.1]

Let $Y_n = X_n - X \to 0$ in L^1. For any fixed $N \in \mathbb{N}$,

$$\sup_n \mathbb{E}[|Y_n| : |Y_n| \geq a] \leq \sup_{n<N} \mathbb{E}[|Y_n| : |Y_n| \geq a] \vee \sup_{n \geq N} \mathbb{E}[|Y_n| : |Y_n| \geq a]$$

$$\therefore \lim_{a \uparrow \infty} \sup_n \mathbb{E}[|Y_n| : |Y_n| \geq a] \leq \sup_{n \geq N} \mathbb{E}[|Y_n|] \to 0 \text{ as } N \to \infty.$$

Thus $\{Y_n\}_{n=1}^{\infty}$ is UI.

$X_n = Y_n + X$ ← $\{X\}$ is UI.

$\therefore \{X_n\}$ is UI

Conversely, suppose $X_n \to_{\mathbb{P}} X$ & $\{X_n\}$ UI.

For $a > 0$, $Y_n \mathbb{1}_{|Y_n| < a} \to 0$ in L^1

$$\|X_n - X\|_{L^1} = \mathbb{E}[|Y_n| \mathbb{1}_{|Y_n| < a}] + \mathbb{E}[|Y_n| \mathbb{1}_{|Y_n| \geq a}]$$

Prop: Let $1 \le p < \infty$ and let $X \in L^p(\Omega, \mathcal{F}, \mathbb{P})$. Then

$$\Lambda = \{ \mathbb{E}_{\mathcal{G}}[X] : \mathcal{G} \subseteq \mathcal{F} \text{ is a sub-}\sigma\text{-field} \}$$

is L^p-bounded, and UI.

Pf. We've shown $\mathbb{E}_{\mathcal{G}}$ is an L^p-contraction i.e. $\|\mathbb{E}_{\mathcal{G}}[X]\|_{L^p} \le \|X\|_{L^p}$

For $p > 1$, it now follows immediately that Λ is UI.

For $p = 1$: we know $|\mathbb{E}_{\mathcal{G}}[X]|$

$\therefore \mathbb{E}[\,|\mathbb{E}_{\mathcal{G}}[X]| : |\mathbb{E}_{\mathcal{G}}[X]| \ge a]$

By Markov's inequality, $\mathbb{P}(|\mathbb{E}_{\mathcal{G}}[X]| \ge a) \le \frac{1}{a} \mathbb{E}[|\mathbb{E}_{\mathcal{G}}[X]|]$

Since $\{X\}$ is UAC, it follows that $\mathbb{E}[|X| : |\mathbb{E}_{\mathcal{G}}[X]| \ge a]$

Cor: If $X_n = \mathbb{E}[X \mid \mathcal{F}_n]$ is a regular martingale, then $\{X_n\}_{n \in \mathbb{N}}$ is UI.

Eg. Let $\{Z_n\}_{n\in\mathbb{N}}$ by iid, $Z_n \overset{d}{=} \frac{1}{2} + \text{Unif}[0,1]$.

We know that $\therefore$ $X_n = Z_0 Z_1 \cdots Z_n$ is a martingale, and $\mathbb{E}[X_n]]$

Is it regular? $X_n \overset{?}{=} \mathbb{E}[X|\mathcal{F}_n]$ for some $X \in L^1$

Notice: $\frac{1}{n} \ln X_n = \frac{1}{n} \sum_{j=0}^{n} \ln Z_j$

$\therefore$ X_n

If X_n were regular, $X_n = \mathbb{E}[X|\mathcal{F}_n]$ for some $X \in L^1$, then $\{X_n\}$ would be UI.

$\therefore$ By Vitali, $X_n \to 0$ in L^1. But then

$\therefore$ $\|X\|_{L^1} = \mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|\mathcal{F}_n]] = \mathbb{E}[X_n]$

We see that L^1-boundedness $\not\Rightarrow$ UI, and there are ≥ 0, L^1-bounded martingales that are not regular.